

E_{11} , generalised space-time and equations of motion in four dimensions

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Abstract

We construct the non-linear realisation of the semi-direct product of E_{11} and its first fundamental representation at low levels in four dimensions. We include the fields for gravity, the scalars and the gauge fields as well as the duals of these fields. The generalised space-time, upon which the fields depend, consists of the usual coordinates of four dimensional space-time and Lorentz scalar coordinates which belong to the 56-dimensional representation of E_7 . We demand that the equations of motion are first order in derivatives of the generalised space-time and then show that they are essentially uniquely determined by the properties of the E_{11} Kac-Moody algebra and its first fundamental representation. The two lowest equations correctly describe the equations of motion of the scalars and the gauge fields once one takes the fields to depend only on the usual four dimensional space-time.

1 Introduction

One of the most remarkable discoveries in the development of supersymmetry, and indeed string theory, was the presence of an E_7 symmetry in the four dimensional maximal supergravity theory [1,2]. There followed the discovery of E_8 [3] and E_9 [4] symmetries in the maximal supergravity theories in three and two dimensions respectively, as well as a conjectured E_{10} symmetry in one dimension [5]. It was also found that the ten dimensional IIB supergravity theory possessed a $SL(2)$ symmetry [6]. Apart from the last symmetry it was universally assumed that these symmetries were a quirk of dimensional reduction on a torus. The one exception was discussed in the papers of references [7, 8]. The first of these papers sacrificed the tangent space group to be just $SO(1,3)$ but was then able to show that eleven dimensional supergravity possessed a $SU(8)$ symmetry; the latter papers in this reference were variations on this theme. While in reference [8] it was argued that eleven dimensional supergravity possessed exceptional structures such as a generalised vielbein associated with the group E_8 , however, the presence of these structures did not lead to the conclusion that the theory possessed an E_8 symmetry group.

Inspired by the observation that the eleven dimensional supergravity theory was a non-linear realisation [9] it was conjectured that the non-linear realisation of E_{11} contained eleven dimensional supergravity [10]. This work was extended to show that the non-linear realisation of E_{11} also contained the IIA [10], IIB [11] and lower dimensional supergravity theories [12,13,14,15]. The different theories result from the different possible decompositions of the Kac-Moody algebra E_{11} that one could take. As the maximal supergravity theories contain all effects at low energy of the underlying theory of strings and branes it was proposed that this underlying theory of strings and branes should possess an E_{11} symmetry [10]. In the early papers our usual notion of space-time was introduced by adjoining the space-time translation generators to the E_{11} algebra in an adhoc step. However, in 2003 the non-linear realisation of the semi-direct product of E_{11} together with generators that belong to its first fundamental representation, denoted l_1 , was considered [16]; this algebra was denoted by $E_{11} \otimes_s l_1$. The highest weight state in the l_1 representations corresponds to the usual space-time translations, but this representation contains an infinite number of elements. We recall that the notion of a semi-direct product is well known to physicists as the Poincaré group is just the semi-direct product of the Lorentz group and the space-time translations. To understand reference [16], that is, the non-linear realisation of $E_{11} \otimes_s l_1$ one has to be familiar with the notion of a non-linear realisation which in this case is quite distinct from what is often called a sigma model. This subject was once well known, at least in some sections of the community and particularly in Russia in the 1960's, but this knowledge seems to have largely been lost in the present, with some notable exceptions. Examples of such non-linear realisations can be found in [9,10,19] and a review of non-linear realisations, and the E_{11} programme, can be found in the book of reference [17]. The non-linear realisation of the type considered in [16] introduces a generalised space-time which is automatically equipped with a generalised vielbein and corresponding generalised tangent space.

The l_1 representation contains an infinite number of elements and so introduces an infinite number of generalised coordinates. Like the adjoint representations of E_{11} , the elements of the l_1 representation can be organised according to the notion of a level [18].

The decomposition which leads to the d dimensional theory, is found by deleting node d in the E_{11} Dynkin diagram and at the lowest level the resulting algebra is $GL(d) \times E_{11-d}$. We recognise the $GL(d)$ algebra as that associated with gravity in d dimensions [19,10] and E_{11-d} as the U duality group in d dimensions. How one finds the d dimensional theory is discussed in detail, for example, in [12-15]. The generalised coordinates that arise from the l_1 representation at the lowest level are the usual coordinates of space-time as well as coordinates which are scalars under the Lorentz group but transform as the 10, $\bar{16}$, $\bar{27}$, 56 and $248 \oplus 1$ of $SL(5)$, $SO(5,5)$, E_6 , E_7 and E_8 for d equal to seven, six, five, four and three dimensions respectively [20,21,13,22]. In fact one can find all the generalised coordinates that are forms, that is, carry completely anti-symmetrised space-time indices. This result for the generators of the l_1 representation, appropriate to d dimensions, are given in the table below [20,21,22]

The form charges in the l_1 representation in d dimensions

D	G	Z	Z^a	$Z^{a_1 a_2}$	$Z^{a_1 \dots a_3}$	$Z^{a_1 \dots a_4}$	$Z^{a_1 \dots a_5}$	$Z^{a_1 \dots a_6}$	$Z^{a_1 \dots a_7}$
8	$SL(3) \otimes SL(2)$	$(\mathbf{3}, \mathbf{2})$	$(\bar{\mathbf{3}}, \mathbf{1})$	$(\mathbf{1}, \mathbf{2})$	$(\mathbf{3}, \mathbf{1})$	$(\bar{\mathbf{3}}, \mathbf{2})$	$(\mathbf{1}, \mathbf{3})$ $(\mathbf{8}, \mathbf{1})$ $(\mathbf{1}, \mathbf{1})$	$(\mathbf{3}, \mathbf{2})$ $(\mathbf{6}, \mathbf{2})$	$(\mathbf{6}, \mathbf{1})$ $(\mathbf{18}, \mathbf{1})$ $(\mathbf{3}, \mathbf{1})$ $(\mathbf{6}, \mathbf{1})$ $(\mathbf{3}, \mathbf{3})$
7	$SL(5)$	$\mathbf{10}$	$\bar{\mathbf{5}}$	$\mathbf{5}$	$\bar{\mathbf{10}}$	$\mathbf{24}$ $\mathbf{1}$	$\mathbf{40}$ $\mathbf{15}$ $\mathbf{10}$	$\mathbf{70}$ $\mathbf{50}$ $\mathbf{45}$ $\mathbf{5}$	- - - -
6	$SO(5, 5)$	$\bar{\mathbf{16}}$	$\mathbf{10}$	$\mathbf{16}$	$\mathbf{45}$ $\mathbf{1}$	$\bar{\mathbf{144}}$ $\mathbf{16}$	$\mathbf{320}$ $\mathbf{126}$ $\mathbf{120}$	- - -	- - -
5	E_6	$\bar{\mathbf{27}}$	$\mathbf{27}$	$\mathbf{78}$ $\mathbf{1}$	$\bar{\mathbf{351}}$ $\mathbf{27}$	$\mathbf{1728}$ $\mathbf{351}$ $\mathbf{27}$	- - -	- - -	- - -
4	E_7	$\mathbf{56}$	$\mathbf{133}$ $\mathbf{1}$	$\mathbf{912}$ $\mathbf{56}$	$\mathbf{8645}$ $\mathbf{1539}$ $\mathbf{133}$ $\mathbf{1}$	- - - -	- - - -	- - - -	- - - -
3	E_8	$\mathbf{248}$ $\mathbf{1}$	$\mathbf{3875}$ $\mathbf{248}$ $\mathbf{1}$	$\mathbf{147250}$ $\mathbf{30380}$ $\mathbf{3875}$ $\mathbf{248}$ $\mathbf{1}$	- - - - -	- - - - -	- - - - -	- - - - -	- - - - -

The corresponding coordinates can be easily read off and they carry the contragredient representations to that of the generators. One sees in the first column the scalar coordinates

mentioned above. The generalised tangent space structure that is inherited from the coordinates is easily read off from the table in an obvious way.

One can view the generalised coordinates from a more physical viewpoint. The l_1 representation can be thought of as containing all the brane charges and so there is a one to one correspondence between the coordinates of the generalised space-time and the brane charges [16,18,23,22,20]. As such one can think of each coordinate as associated with a given type of brane probe. Furthermore for every generator in the Borel subalgebra of E_{11} there is a corresponding element in the l_1 representation [18], and as a result for every field in the non-linear realisation one finds a corresponding coordinate. For example, in eleven dimensions at lowest level one has the usual field of gravity $h_a{}^b$ associated with which one has the usual coordinate x^a of space-time, at the level one we find the three form field $A_{a_1 a_2 a_3}$ with associated coordinate $x_{a_1 a_2}$, at level two we have the six form field $A_{a_1 \dots a_6}$ with the associated coordinate $x_{a_1 \dots a_5}$, at level three the dual field of gravity $h_{a_1 \dots a_8, b}$ with a corresponding coordinate $x_{a_1 \dots a_7, b}$ and similarly at higher levels [16]. One can think of this as generalisation of the notion of space-time introduced by Einstein that takes into account the presence of fields, required by supersymmetry, in addition to the metric. The precise correspondence, for the four dimensional theory, between the fields and the coordinates can be found later in this paper.

Although quite a number of the predictions of E_{11} have been verified, see for example [13,17] for an account, the radically new nature of the generalised space-time has, until relatively recently, discouraged the systematic calculation of the $E_{11} \otimes_s l_1$ non-linear realisation. In the early papers on E_{11} only the coordinate x^a was used and the symmetries of the non-linear realisation were only implemented at lowest levels. This particularly, applies to the local subalgebra which plays an important part in the non-linear realisation. The local subalgebra is taken to be the Cartan involution invariant subalgebra and it was usually taken to be just that at the lowest level which, in eleven dimensions, is just the Lorentz algebra. As a result much of the power of the non-linear realisation was lost. Nonetheless many of the features of the supergravity theories were recovered. With retrospect one can view the early attempts to construct the dynamics, see for example [10,11], as using the $E_{11} \otimes_s l_1$ non-linear realisation but only keeping the lowest level l_1 generators, which, in eleven dimensions, are just the usual space-time translations and so only the usual coordinates of space-time.

One of the first papers to use some of the higher level generalised coordinates is given in reference [13] which was used to construct all gauged supergravities in five dimensions [13], a result not previously known. In this construction some of the generalised coordinates and their corresponding components of the generalised vielbein played an important role. However, the remaining coordinates of the l_1 representations were discarded and the generalised space-time that remained was a slice taken in the l_1 representations and E_{11} . The techniques used in this paper could easily be applied to find all gauged supergravities in all dimensions.

The construction of the $E_{11} \otimes_s l_1$ non-linear realisation at lowest level in four dimensions was carried out in reference [24, 25]. This contained the usual coordinates of space-time and Lorentz scalar coordinates which belonged to the 56-dimensional representation of E_7 , mentioned above, that is the content of the l_1 representations at lowest level.

It also took the fields in the l_1 representation at lowest level that is the metric and the scalar fields. Much of these papers were devoted to the part of the theory that lives on the 56-dimensional space. In contrast to the equation of motion approach pursued in most E_{11} papers, this paper constructed an invariant Lagrangian. This Lagrangian was not uniquely determined by the symmetries of the non-linear realisation, which were taken to be those at lowest level, and it contained several undetermined constants. However, it was realised in the papers of reference [24,25] that if one restricted the dependence of the fields to be only on the usual coordinates of space-time then, for a suitable choice of the constants, the action was gauge and general coordinate invariant. More recently, the $E_{11} \otimes_s l_1$ non-linear realisations, at lowest level, in the l_1 representation and in E_{11} , and also discarding the fields and coordinates of the usual space-time, were constructed in dimensions four to seven and corresponding Lagrangians were constructed [26]. These Lagrangians in six and seven dimensions had previously been constructed [27] using the coordinates introduced into the first quantised dynamics with the aim of encoding duality symmetries [28]. However, it was apparent [26] that these were just the result of the non-linear realisation of $E_{11} \otimes_s l_1$ at lowest level which was then further truncated in the way just mentioned.

One of the first papers to compute the $E_{11} \otimes_s l_1$ non-linear realisation at higher levels and also keeping some of the higher level symmetries was contained in references [29,30] which computed the ten dimensional IIA theory. It kept the E_{11} fields at levels zero and one which contained the fields of the NS-NS and R-R sectors of the IIA string respectively, with the coordinates of the l_1 representation at level zero. The latter consisted of the usual coordinates of ten dimensional space-time as well as coordinates which were all forms of odd rank. Considering quantities that were first order in the derivatives of the generalised space-time it was shown that there existed only two covariant objects, both of which were uniquely determined and transformed into themselves. Setting one of these to zero resulted in a set of equations of motion which were those of type IIA supergravity when the fields were taken to depend only on the usual coordinates of ten dimensional space-time. The way the results in these papers was phrased were a bit different but it is equivalent to the statement just made.

Very recently the non-linear realisation of $E_{11} \otimes_s l_1$ in eleven dimensions was constructed keeping the fields of gravity, three form, six form and dual gravity fields and the usual coordinates of space-time as well as the two form and five form coordinates [31]. In other words, this calculation kept the coordinates of the l_1 representation and E_{11} up to and including levels two and three respectively. As a result one could impose the higher level symmetries contained in the $E_{11} \otimes_s l_1$ non-linear realisation and, in particular, the local symmetries. As in ten dimensions we considered quantities that were first order in the derivatives of the generalised space-time and again found that there existed only two such covariant objects both of which were unique and transformed into themselves. Setting one of them to zero resulted in a set of equations of motion, the first of which correctly described the equation of motion for the three form and six form fields when the dependence on the higher level coordinates and fields beyond the dual graviton were discarded. The second equation in this set described the equation of motion relating the usual field of gravity and the dual gravity field. This equation was very close to being correct and one source of the discrepancy may be accounted for by missing contributions from higher

order fields which were omitted. We note that should one succeed in finding an equation that correctly describes gravity then the E_{11} conjecture would be confirmed, that is, the non-linear realisation of $E_{11} \otimes_s l_1$ is an extension of the maximal supergravity theories. In this event one would have to take seriously the generalised space-time associated with the l_1 representation.

In this paper we will carry out a similar calculation but in four dimensions. That is we will carry out the $E_{11} \otimes_s l_1$ non-linear realisation in the decomposition that leads to the four-dimensional theory. We will keep the fields in the l_1 representation up to level four, that is we include the fields of gravity, scalars, one forms, two forms and the dual field of gravity. From the l_1 representation we will take the usual coordinates of space-time and the Lorentz scalar coordinates that transform in the 56-dimensional representations of E_7 , that is we only keep the level zero part of the l_1 representation. However, we will do the calculation in such a way that we will impose some of the crucial higher level symmetries of the non-linear realisation. The $E_{11} \otimes_s l_1$ algebra for the eleven dimensional theory is relatively simple and well studied up to the required levels however, the complicated index structure of the fields means that the calculation of the equations of motion and the verification of their invariance, given in [31], is rather intricate. Although the four dimensional theory has more fields compared to the eleven dimensional theory they have a much simpler index structure. Hence although it is much more complicated to find the $E_{11} \otimes_s l_1$ algebra for the required decomposition, the calculation of the equations of motion is much simpler. To find the four dimensional theory one must decompose E_{11} and the l_1 representations into representations of $GL(4) \otimes E_7$. While this is relatively straightforward, in order to correctly implement even the lowest order local subalgebra of the non-linear realisation one must then further decompose into representations of $SO(1, 3) \otimes SU(8)$. This complicated calculation takes up much of this paper, however, there are many checks one can carry out to verify that the results found are correct. One further advantage of four dimensions is that the usual gravity field and the dual gravity field, and their related coordinates, have similar index structures. As such one is hopefully in a better position to resolve the problems associated with the dual gravity field.

We will consider how objects that are first order in derivatives with respect to the generalised space-time transform under the symmetries of the non-linear realisation. We will show that there are two sets of objects that transform into themselves and are uniquely specified by the symmetries. Each set contains an infinite number of objects and setting one set to zero leads at the lowest level to the correct equations of motion for the scalars and gauge fields provided we take the fields to only depend on the coordinates of the usual four dimensional space-time. We will also give some preliminary results on the higher level fields.

None of the papers mentioned above gives a satisfactory account of how the familiar space-time we are used to emerges naturally from the generalised space-time, or put another way, how does one discard most of the coordinates of the l_1 representation to find the theories we are familiar with. This dilemma is considered further in the discussion section.

Before concluding the introduction we will make a few remarks on the relation of the E_{11} programme to other approaches. We begin by contrasting the usual view of M theory with the E_{11} conjecture. It is often stated that all the different string theories are different

aspects of an eleven dimensional theory, called M theory. Of course we do not know much about M theory so the meaning of this statement is not clear. In contrast all the maximal supergravity theories can be obtained from the $E_{11} \otimes_s l_1$ non-linear realisation by taking different decompositions. As such from the E_{11} viewpoint all the different theories are on an equal footing and indeed are dual to each other, being different descriptions of the same underlying theory, and indeed no space-time dimension is preferred. The mapping between the different theories is for example discussed in reference [23].

Subsequent to the 2003 proposal of reference [16] a number of other approaches involving some kind of generalised geometry have been considered. The most popular is called doubled geometry, see [32] and references therein. This approach was inspired by earlier papers that considered in addition to the usual coordinate of space-time x^a the coordinate y_a and constructed an $O(10,10)$, T duality, invariant ten dimensional theory. This theory had the same fields as the NS-NS sector of the superstring but they depended on both of the just mentioned coordinates. However, the result was none other than the $E_{11} \otimes_s l_1$ non-linear realisation suitable for the IIA theory at lowest level [29]. The advantage of viewing it this way is that it is part of a much large conceptual framework where all the symmetries are automatically encoded. Indeed the non-linear realisation of $E_{11} \otimes_s l_1$ at level one leads to the inclusion of the R-R sector fields [30]. The level zero and one calculations of the non-linear realisation appropriate for the IIA theory are very simple and can be performed in a few pages without the need for any guess work. The one point contained in the literature on doubled field theory which does not follow from the $E_{11} \otimes_s l_1$ non-linear realisation is how to discard the y_a coordinates. Rather than just discard them as in references [25,26] they adopt what is called a section condition, however, in practice it seems there is not so much difference.

There is yet another approach inspired by the work of references [33] and [34]. This introduced an extended tangent space, associated with $O(D,D)$ but does not extend our usual notion of space-time, see for example [35] and references therein. In this approach one does not try to find additional symmetries, but rather packages the theory up into a generalised geometry. In ten dimensions the tangent space is essentially doubled compared to that taken conventionally, however this is precisely the same tangent space as arises in the non-linear realisation of $E_{11} \otimes_s l_1$ appropriate to ten dimensions at lowest level [29,30]. While in lower dimensions the tangent spaces and tangent groups are just those contained in the $E_{11} \otimes_s l_1$ non-linear realisation at lowest level [20,21,13,22]. While there has not been a detailed study to investigate the connection to the non-linear realisation of $E_{11} \otimes_s l_1$ it would seem inevitable that it is just the non-linear realisation of $E_{11} \otimes_s l_1$ with the l_1 part taken to be just the usual coordinates of space-time and the E_{11} part up to the level that incorporates all the usual supergravity fields. Thus, while the works of reference [35] try to generalise Einstein geometry taking into account global considerations, the required structures are very likely to be automatically encoded in the $E_{11} \otimes_s l_1$ non-linear realisation.

2 The E_{11} algebra and l_1 representation viewed from four dimensions

In this section we will formulate the E_{11} algebra and the l_1 representation in such a way that their non-linear realisation leads to a theory in four dimensions. We will begin by finding their decompositions in terms of representations of $GL(4) \otimes E_7$ rather than the more common $SL(11)$ decomposition which leads to the eleven dimensional theory. An important

role in the construction of the non-linear realisation, that is the dynamics, is played by the Cartan involution invariant subalgebra of E_{11} , denoted $I_c(E_{11})$. At lowest level this is just the subalgebra $SO(1,3) \otimes SU(8)$. The $SU(8)$ factor is just the Cartan involution invariant subgroup of E_7 while $SO(1,3)$ is the Cartan involution invariant subalgebra of $SL(4)$; these are the same as the respective maximal compact subgroup for the real forms of E_7 and A_3 with which we are working. As such we need a formulation of E_{11} and the l_1 representation in which the subalgebra $I_c(E_{11})$ is apparent and in particular the subalgebra $SO(1,3) \otimes SU(8)$. While it is obvious how representations of $SL(4)$ can be rewritten in terms of $SO(1,3)$ this is not quite so clear for the $SU(8)$ hidden within E_7 . This problem has been well studied in the mathematics literature and an account for physicists can be found in appendix B of reference [2]. As explained in this reference the best way to find this $SU(8)$ subalgebra of E_7 is to first identify the more obvious $SL(8)$ subgroup of E_7 and then decompose the adjoint representations of E_7 in terms of representations of this $SL(8)$. The desired $SU(8)$ subalgebra, and the decomposition of the adjoint representation of E_7 into representations of it, can then be constructed. We note that the $SL(8)$ and $SU(8)$ have in common their obvious $SO(8)$ subgroups, but they are not different real forms of the same subalgebra of E_7 when viewed in its complex form.

For the calculations in this paper we must generalise this results to find, at low levels, the Cartan involution subalgebra of E_{11} , that is $I_c(E_{11})$, and then decompose E_{11} , and its l_1 representations, into representations of $I_c(E_{11})$. This is the task of section two. At lowest level this was carried out in reference [26], that is, the $SU(8)$ contained within E_{11} was identified and the E_7 and 56-dimensional representation in l_1 were decomposed in terms of this $SU(8)$.

2.1 The $GL(4) \otimes E_7$ decomposition of the E_{11} algebra and the l_1 representation

To find the four-dimensional theory from the non-linear realisation of $E_{11} \otimes_s l_1$ we delete the fourth node in the E_{11} Dynkin diagram, see figure 2.1, and consider the decomposition of $E_{11} \otimes_s l_1$ into representations of the subalgebra that results, that is, $GL(4) \otimes E_7$.

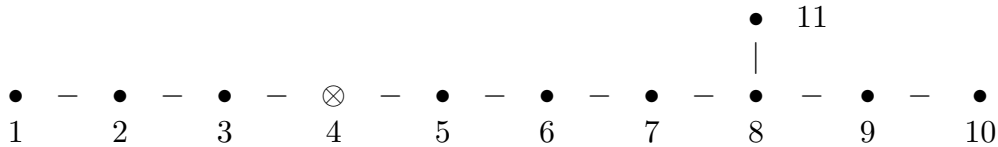


Figure 2.1: The E_{11} Dynkin diagram from the viewpoint of the four-dimensional theory.

The $GL(4)$ factor, whose generators we denote by K^a_b , $a, b = 1, \dots, 4$ leads in the non-linear realisation to the familiar field used to describe gravity and the E_7 , whose generator we denote by R^α , is the well known symmetry group in four dimensions whose corresponding fields in the non-linear realisation are the seventy scalars.

The representations that occur in this decomposition can be classified according to a level. In this case the level is just the number of upper minus lower $GL(4)$ indices that the generator possess. Thus the level zero E_{11} generators are just $GL(4) \otimes E_7$. The interested

reader can find a more formal account of the level in earlier E_{11} papers and in the book [17]. Ordered by their level the decomposition of E_{11} into representations of $GL(4) \otimes E_7$ is given by

$$\begin{aligned} &K^a_b(15, 1, 0), R^\alpha(1, 133, 0); R^{aN}(4, 56, 1); R^{a_1 a_2 \alpha}(6, 133, 2), \\ &\hat{K}^{ab}(10, 1, 2); R^{a_1 a_2 a_3 \lambda}(4, 912, 3), R^{a_1 a_2, bN}(20, 56, 3); \dots \end{aligned} \quad (2.1.1)$$

where \dots indicate generators at level four and above. The first two figures in the brackets indicate the dimensions of the $SL(4)$ and E_7 representations respectively, while the last figure is the level. We have not displayed the negative level generators, except for those at level zero, however, they have the same index structure except that their indices are now subscripts. The $GL(4)$ indices are given as a, b, a_1, a_2, \dots . The indices on the generators $R^{a_1 a_2 \alpha}$ and $R^{a_1 a_2 a_3 \lambda}$ are totally anti-symmetrised. The generator \hat{K}^{ab} is subject to the condition $\hat{K}^{ab} = \hat{K}^{(ab)}$ and the generator $R^{a_1 a_2, bN}$ satisfies $R^{[a_1 a_2, b]N} = 0$.

As explained later, in the non-linear realisation there is a one to one correspondence between the fields that arise and the generators of the Borel subalgebra, that is those of level zero and positive level. The $GL(4) \otimes E_7$ indices on the fields are inherited from those on these generators, for example $R^{aN} A_{aN}$. As such, the generators with completely anti-symmetrised indices, the so called form generators, lead to the gauge fields $A_{aN}, A_{a_1 a_2 \alpha}, A_{a_1 a_2 a_3 \lambda}$. One expects, in the resulting dynamics, that the first gauge fields will satisfy some kind of self-duality condition, the second gauge fields are dual to the scalars and the third lead to a cosmological constant and so classify the gauged supergravities [12,36]. As we have mentioned the usual field h_a^b of gravity corresponds to the generator K^a_b while the generator \hat{K}^{ab} leads to the dual gravity field, denoted by \hat{h}_{ab} . In this paper we will be only concerned with the generators up to, and including, level two.

We also decompose the l_1 representation into representations of $GL(4) \otimes E_7$. Listed according to their level the results is

$$\begin{aligned} &P_a(4, 1, 0), Z^N(1, 56, 1), Z^{a\alpha}(4, 133, 2), Z^a(4, 1, 2), Z^{a_1 a_2 N}(6, 56, 3), \\ &Z^{(a_1 a_2)N}(10, 56, 3), Z^{a_1 a_2 \lambda}(6, 912, 3), \dots \end{aligned} \quad (2.1.2)$$

where \dots denoted objects at level four and above. In the non-linear realisation these lead to the generalised space-time which has the corresponding coordinates

$$x^a, x_N, x_{a\alpha}, \hat{x}_a, x_{a_1 a_2 N}, x_{(a_1 a_2)N}, x_{a_1 a_2 \lambda}, \dots \quad (2.1.3)$$

We note that for the four dimensional theory, the gravity and dual gravity fields and usual coordinates x^a and dual gravity coordinates \hat{x}_a appear on a much more symmetrical footing than in other dimensions.

One can find the generators of equation (2.1.1) and (2.1.2) by simply "dimensionally reducing" to four dimensions the E_{11} algebra, and the l_1 representation when written in their eleven dimensional formulations. From the group theoretic view point this is equivalent to decomposing the $SL(11)$ formulation of the E_{11} algebra and the l_1 representation

into representations of $GL(4) \otimes SL(7)$. The $SL(11)$ in question is the algebra that result from deleting node eleven in the E_{11} Dynkin diagram of figure 2.1 and so its Dynkin diagram consists of nodes one to ten. We recall that the $GL(4) \otimes E_7$ subalgebra arose from deleting node four in the Dynkin diagram of figure 2.1 and the $SL(4)$ and $SL(7)$ subalgebras correspond to the Dynkin diagrams which consist of nodes one to three and nodes five to ten respectively. Having found the decomposition into $GL(4) \otimes SL(7)$ representations we can then repackage the $SL(7)$ representations into those of E_7 . The eleven dimensional, or $SL(11)$, formulation of the E_{11} algebra is well known, and when listed according to the appropriate level, it contains the positive level generators [10,37]

$$\begin{aligned} & K^{\hat{a}}_{\hat{b}}(0), R^{\hat{a}_1 \hat{a}_2 \hat{a}_3}(1), R^{\hat{a}_1 \dots \hat{a}_6}(2), R^{\hat{a}_1 \dots \hat{a}_8, \hat{b}}(3), \\ & R^{\hat{a}_1 \dots \hat{a}_9, \hat{b}_1 \hat{b}_2 \hat{b}_3}(4), R^{\hat{a}_1 \dots \hat{a}_{11}, \hat{b}}(4), R^{\hat{a}_1 \dots \hat{a}_{10}, (\hat{b}_1 \hat{b}_2)}(4), \dots \end{aligned} \quad (2.1.4)$$

where $\hat{a}, \hat{b} = 1, \dots, 11$. In this equation \dots denotes generators at level five and higher. The generators obey irreducibility conditions such as $R^{[\hat{a}_1 \dots \hat{a}_8, \hat{b}]} = 0, \dots$. The numbers in the brackets indicate the level appropriate to the $SL(11)$ decomposition. This level arises from deleting node eleven and it is different to the level discussed above that is associated with deleting node four. The eleven dimensional level can be thought of as just the number of up minus down eleven dimensional indices divided by three.

The l_1 representation, listed according to increasing level, contains [16,18,20]

$$\begin{aligned} & P_{\hat{a}}(0), Z^{\hat{a}_1 \hat{a}_2}(1), Z^{\hat{a}_1 \dots \hat{a}_5}(2), Z^{\hat{a}_1 \dots \hat{a}_7, b}(3), Z^{\hat{a}_1 \dots \hat{a}_8}(3), \\ & Z^{\hat{b}_1 \hat{b}_2 \hat{b}_3, \hat{a}_1 \dots \hat{a}_8}(4), Z^{(\hat{c} \hat{d}), \hat{a}_1 \dots \hat{a}_9}(4), Z^{\hat{c} \hat{d}, \hat{a}_1 \dots \hat{a}_9}(4), Z^{\hat{c}, \hat{a}_1 \dots \hat{a}_{10}}(4), Z(4) \\ & Z^{\hat{c}, \hat{d}_1 \dots \hat{d}_4, \hat{a}_1 \dots \hat{a}_9}(5), Z^{\hat{c}_1 \dots \hat{c}_6, \hat{a}_1 \dots \hat{a}_8}(5), Z^{\hat{c}_1 \dots \hat{c}_5, \hat{a}_1 \dots \hat{a}_9}(5), \\ & Z^{\hat{d}_1, \hat{c}_1 \hat{c}_2 \hat{c}_3, \hat{a}_1 \dots \hat{a}_{10}}(5), Z^{\hat{c}_1 \dots \hat{c}_4, \hat{a}_1 \dots \hat{a}_{10}}(5, -2), Z^{(\hat{c}_1 \hat{c}_2, \hat{c}_3)}(5), Z^{\hat{c}, \hat{a}_1 \hat{a}_2}(5), \dots \end{aligned} \quad (2.1.5)$$

These generators satisfy irreducibility conditions such as $Z^{[\hat{a}_1 \dots \hat{a}_7, \hat{b}]} = 0, \dots$

To carry out the decomposition from representations of $SL(11)$ into those of $GL(4) \otimes SL(7)$ we divide the values of the indices \hat{a}, \hat{b}, \dots into the ranges, one to four and the remainder, that is, five to eleven, which we denote by the labels a and i respectively. Carrying this out for the l_1 representation given in equation (2.1.5) we find that

$$P_a; \dot{P}_i, \dot{Z}^{i_1 i_2}, \dot{Z}^{i_1 \dots i_5}, \dot{Z}^{i_1 \dots i_7, j}, \dot{Z}^{ai}, Z^{ai_1 \dots i_4}, \dots \quad (2.1.6)$$

We have placed a dot on the resulting objects as we will use the symbols $Z^{i_1 i_2}$ etc for a later purpose. We now have to assemble these into representations of $GL(4) \otimes E_7$. Those of $GL(4)$ are the same, but we must assemble the representations of $SL(7)$ into those of E_7 . For example, between the first semi-colon and the second, we recognise the $56 = 7 + 21 + 21 + 7$ -dimensional representation of E_7 contained in the Z^N of equation (2.1.2).

Proceeding in a similar way we can decompose the generators of the E_{11} algebra of equation (2.1.4) into representations of $SL(7)$ to find

$$K^a{}_b; \dot{K}^i{}_j, \dot{R}^{i_1 i_2 i_3}, \dot{R}_{i_1 i_2 i_3}, \dot{R}^{i_1 \dots i_6}, \dot{R}_{i_1 \dots i_6}; \dot{K}^a{}_i, \dot{R}^{a i_1 i_2}, \dot{R}^{a i_1 \dots i_5}, \dot{R}^{i_1 \dots i_7, a} : \dots \quad (2.1.7)$$

The first generators before the first semicolon are those of the $15 + 1$ -dimensional representation of $GL(4)$. The generators between the next semicolons are those of the adjoint representations of E_7 corresponding to the decomposition $133 = 48 + 1 + 35 + 35 + 7 + 7$ and those after the next semicolon belong to the $56 = 7 + 21 + 21 + 7$ -dimensional representation of E_7 ; thus we find agreement with equation (2.1.1).

2.2 The $SL(8)$ Formulation for the E_{11} algebra and l_1 representation

As we have explained at the beginning of section two an important role in the construction of the non-linear realisation is played by the Cartan involution invariant subalgebra of E_{11} which at lowest level is the subalgebra $SO(1, 3) \otimes SU(8)$. To locate this algebra we first identify the $SL(8)$ subgroup of E_7 whose generators we denote by $K^I{}_J, I, J = 1, \dots, 8$. The obvious $SL(7)$ subgroup of $SL(8)$ is the $SL(7)$ subgroup discussed in the last section. In terms of the generators of equation (2.1.7) the generators of $SL(8)$ are given by [26]

$$K^i{}_j = \tilde{K}^i{}_j - \frac{1}{6} \delta_j^i \sum_k \tilde{K}^k{}_k, \quad i, j = 1, \dots, 7$$

$$K^8{}_j = -\frac{2}{6!} \epsilon_{j i_1 \dots i_6} \dot{R}^{i_1 \dots i_6}, \quad K^j{}_8 = \frac{2}{6!} \epsilon^{j i_1 \dots i_6} \dot{R}_{i_1 \dots i_6}. \quad (2.2.1)$$

where

$$\tilde{K}^i{}_j = \dot{K}^i{}_j - \frac{1}{2} \delta_j^i \sum_a K^a{}_a \quad (2.2.2)$$

On the right hand side of these equations the symbols $\dot{K}^i{}_j, \dot{R}^{i_1 \dots i_6}$ and $\dot{R}_{i_1 \dots i_6}$ are those found by the decomposition given in equation (2.1.7) and should not be confused with the $K^i{}_j$ on the left hand side of equation (2.2.1) which are part of the $SL(8)$ generators. It is straightforward to verify, using the E_{11} algebra given in appendix A, that they do indeed satisfy the $SL(8)$ algebra, that is,

$$[K^I{}_J, K^L{}_M] = \delta_J^L K^I{}_M - \delta_M^I K^L{}_J \quad (2.2.3)$$

Since we are dealing with $SL(8)$ the generators are traceless and so $K^8{}_8 = -\sum_{i=1}^7 K^i{}_i$

The remaining generators of E_7 belong to the seventy-dimensional representation of $SL(8)$ and are contained in the generator $R^{I_1 \dots I_4}$ whose indices are totally antisymmetric. In terms of the decomposed generators of equation (2.1.7) these are given by

$$R^{i_1 i_2 i_3 8} = \frac{1}{12} \dot{R}^{i_1 i_2 i_3}, \quad R^{i_1 \dots i_4} = \frac{1}{12 \cdot 3!} \epsilon^{i_1 \dots i_4 j_1 j_2 j_3} \dot{R}_{j_1 j_2 j_3} \quad (2.2.4)$$

The commutators of these generators with those of $SL(8)$ are given by

$$[K^I{}_J, R^{L_1 \dots L_4}] = 4 \delta_J^{[L_1} R^{L_2 \dots L_4]} - \frac{1}{2} \delta_J^I R^{L_1 \dots L_4} \quad (2.2.5)$$

while the remaining E_7 commutators are given by

$$\begin{aligned} [R^{I_1 \cdots I_4}, R^{J_1 \cdots J_4}] &= -\frac{1}{8} \{ \epsilon^{J_1 \cdots J_4 L [I_1 \cdots I_3 K^{I_4}]_L} - \epsilon^{I_1 \cdots I_4 L [J_1 \cdots J_3 K^{J_4}]_L} \} \\ &= -\frac{1}{4} \epsilon^{J_1 \cdots J_4 L [I_1 \cdots I_3 K^{I_4}]_L}. \end{aligned} \quad (2.2.6)$$

In deriving the last relation we have used the identity

$$\epsilon^{J_1 \cdots J_4 L [I_1 \cdots I_3 S^{I_4}]_L} + \epsilon^{I_1 \cdots I_4 L [J_1 \cdots J_3 S^{J_4}]_L} = -\epsilon^{J_1 \cdots J_4 I_1 \cdots I_4} \sum_N S^N_N \quad (2.2.7)$$

valid for any object S^I_J . This identity is easily proved by taking values for the indices.

Proceeding in a similar way one finds that the positive level generators of the E_{11} algebra, when written in terms of $GL(4) \otimes SL(8)$ representations, take the form

$$\begin{aligned} K^I_J(1, 63), R^{I_1 \cdots I_4}(1, 70), K^a_b(16, 1); R^{a I_1 I_2}(4, 28), R^a_{I_1 I_2}(4, \bar{28}); \\ R^{a_1 a_2 I}_J(6, 63), R^{a_1 a_2 I_1 \cdots I_4}(6, 70), \hat{K}^{ab}(10, 1), \dots \end{aligned} \quad (2.2.8)$$

The two numbers in brackets give the dimensions of their $SL(4)$ and $SL(8)$ representations respectively. The negative definite level generators are given by

$$\tilde{R}_{a I_1 I_2}(4, \bar{28}), \tilde{R}_a^{I_1 I_2}(4, 28); \tilde{R}_{a_1 a_2}^I(6, 63), \tilde{R}_{a_1 a_2}^{I_1 \cdots I_4}(6, 70), \tilde{K}_{ab}(10, 1), \dots \quad (2.2.9)$$

The level one generators of equation (2.2.8) are identified with the underlying E_{11} algebra of equation (2.1.7) as follows

$$\begin{aligned} R^{a i_1 i_2} &= \dot{R}^{a i_1 i_2}, \quad R^{a j 8} = \frac{2}{7!} \epsilon_{i_1 \cdots i_7} \dot{R}^{a i_1 \cdots i_7, j} \\ R^a_{i_1 i_2} &= -\frac{2}{5!} \epsilon_{i_1 i_2 j_1 \cdots i_5} \dot{R}^{a j_1 \cdots j_5}, \quad R^a_{i 8} = -\dot{K}^a_i \end{aligned} \quad (2.2.10)$$

while for the level minus one generators the identification is given by

$$\begin{aligned} \tilde{R}_{a i_1 i_2} &= \dot{R}_{a i_1 i_2}, \quad \tilde{R}_{a j 8} = \frac{2}{7!} \epsilon^{i_1 \cdots i_7} \dot{R}_{a i_1 \cdots i_7, j} \\ \tilde{R}^{a i_1 i_2} &= -\frac{2}{5!} \epsilon^{i_1 i_2 j_1 \cdots i_5} \dot{R}_{a j_1 \cdots j_5}, \quad \tilde{R}_a^{i 8} = \dot{K}^i_a \end{aligned} \quad (2.2.11)$$

Using the above identifications and the E_{11} commutators given in appendix A one can deduce that the commutators of the E_{11} algebra when written in terms of the $GL(4) \otimes SL(8)$ decomposition, that is, as given in equations (2.2.8) and (2.2.9). The commutators with the $SL(8)$ generators of E_7 are given by

$$[K^I_J, R^{a K_1 K_2}] = 2\delta_J^{[K_1]} R^{a I | K_2]} - \frac{1}{4} \delta_J^I R^{a K_1 K_2} \quad [K^I_J, R^a_{K_1 K_2}] = -2\delta_{[K_1]}^I R^a_{J | K_2]} + \frac{1}{4} \delta_J^I R^a_{K_1 K_2} \quad (2.2.12)$$

together with analogous result for the other generators. The commutators with the $R^{I_1 \dots I_4}$ generators of E_7 are

$$\begin{aligned}
[R^{I_1 \dots I_4}, R^{a_{J_1 J_2}}] &= \frac{1}{4!} \epsilon^{I_1 \dots I_4 J_1 J_2 K_1 K_2} R^a_{K_1 K_2} \quad [R^{I_1 \dots I_4}, R^a_{J_1 J_2}] = \delta_{J_1 J_2}^{[I_1 I_2} R^{a I_3 I_4]} \\
[R^{I_1 \dots I_4}, R^{a_1 a_2 J}_{K}] &= -4\delta_K^{[I_1} R^{a_1 a_2 J | I_2 I_3 I_4]} + \frac{1}{2} \delta_K^J R^{a_1 a_2 I_1 \dots I_4}, \\
[R^{I_1 \dots I_4}, \tilde{R}_{a_1 a_2}^J{}_{K}] &= 4\delta_{[I_1}^K \tilde{R}_{a_1 a_2 J | I_2 I_3 I_4]} - \frac{1}{2} \delta_K^J \tilde{R}_{a_1 a_2 I_1 \dots I_4}, \\
[R^{I_1 \dots I_4}, R^{a_1 a_2 J_1 \dots J_4}] &= \frac{1}{36} \epsilon^{I_1 \dots I_4 L [J_1 J_2 J_3 | R^{a_1 a_2 | J_4]}_L = -\frac{1}{36} \epsilon^{J_1 \dots J_4 L [I_1 I_2 I_3 | R^{a_1 a_2 | I_4]}_L \\
[R^{I_1 \dots I_4}, \tilde{R}_{a_1 a_2 J_1 \dots J_4}] &= -\frac{2}{3} \delta_{[J_1 J_2 J_3}^{[I_1 I_2 I_3} \tilde{R}_{a_1 a_2}{}^{I_4]}_{J_4]} \quad (2.2.13)
\end{aligned}$$

The commutators of the level one generators with themselves are given by

$$\begin{aligned}
[R^{a I_1 I_2}, R^{b I_3 I_4}] &= -12 R^{ab I_1 \dots I_4}, \quad [R^{a I_1 I_2}, R^b_{J_1 J_2}] = +4\delta_{[J_1}^{[I_1} R^{a b | I_2]}_{J_2]} + 2\delta_{J_1 J_2}^{I_1 I_2} \hat{K}^{ab} \\
[R^a_{I_1 I_2}, R^b_{J_1 J_2}] &= \frac{1}{2} \epsilon_{I_1 I_2 J_1 J_2 K_1 \dots K_4} R^{a_1 a_2 K_1 \dots K_4} = 12 \star R^{a_1 a_2}_{I_1 I_2 J_1 J_2} \quad (2.2.14)
\end{aligned}$$

where $\star R^{a_1 a_2}_{I_1 \dots I_4} = \frac{1}{4!} \epsilon_{I_1 \dots I_4 J_1 \dots J_4} R^{a_1 a_2 J_1 \dots J_4}$. While the equivalent commutators for the level minus one generators with themselves are given by

$$\begin{aligned}
[\tilde{R}_{a I_1 I_2}, \tilde{R}_{b J_1 J_2}] &= -12 \star \tilde{R}_{ab I_1 I_2 J_1 J_2}, \quad [\tilde{R}_{a I_1 I_2}, \tilde{R}_b^{J_1 J_2}] = +4\delta_{[I_1}^{J_1} \tilde{R}_{ab}^{J_2]}_{I_2]} + 2\delta_{I_1 I_2}^{J_1 J_2} \tilde{\hat{K}}_{ab} \\
[\tilde{R}_a^{I_1 I_2}, \tilde{R}_b^{J_1 J_2}] &= 12 \tilde{R}_{ab}^{I_1 I_2 J_1 J_2} \quad (2.2.15)
\end{aligned}$$

The commutators between the level one and minus one generators are given by

$$\begin{aligned}
[R^{a I_1 I_2}, \tilde{R}_{b J_1 J_2}] &= 2\delta_{J_1 J_2}^{I_1 I_2} K_b^a + 4\delta_b^a \delta_{[J_1}^{[I_1} K^{I_2]}_{J_2]} - \delta_b^a \delta_{J_1 J_2}^{I_1 I_2} \sum_{c=1}^4 K^c_c \\
[R^{a I_1 I_2}, \tilde{R}_b^{J_1 J_2}] &= -12\delta_b^a R^{I_1 I_2 J_1 J_2}, \quad [R^a_{I_1 I_2}, \tilde{R}_{b J_1 J_2}] = 12\delta_b^a \star R_{I_1 I_2 J_1 J_2} \\
[R^a_{I_1 I_2}, \tilde{R}_b^{J_1 J_2}] &= -2\delta_{I_1 I_2}^{J_1 J_2} K_b^a + 4\delta_b^a \delta_{[I_1}^{[J_1} K^{J_2]}_{I_2]} + \delta_b^a \delta_{I_1 I_2}^{J_1 J_2} \sum_{c=1}^4 K^c_c \quad (2.2.16)
\end{aligned}$$

Finally we list the commutators between the level two and minus one generators.

$$\begin{aligned}
[R^{ab I}_{J}, \tilde{R}_{c K_1 K_2}] &= -4\delta_c^{[a} \delta_{[K_1}^I R^{b]}_{J | K_2]} + \frac{1}{2} \delta_c^{[a} \delta_J^I R^{b]}_{K_1 K_2}, \\
[R^{ab I}_{J}, \tilde{R}_c^{K_1 K_2}] &= 4\delta_c^{[a} \delta_J^{[K_1} R^{b]}_{I | K_2]} - \frac{1}{2} \delta_c^{[a} \delta_J^I R^{b]}_{K_1 K_2}
\end{aligned}$$

$$[R^{abI_1 \dots I_4}, \tilde{R}_{cK_1K_2}] = 2\delta_c^{[a} \delta_{K_1K_2}^{I_1I_2} R^{b]I_3I_4}, \quad [R^{abI_1 \dots I_4}, \tilde{R}_c^{K_1K_2}] = \frac{1}{12} \epsilon^{I_1 \dots I_4 K_1 K_2 J_1 J_2} \delta_c^{[a} R^{b] J_1 J_2} \quad (2.2.17)$$

$$[\hat{K}^{ab}, \tilde{R}_{cJ_1J_2}] = -\delta_c^{[a} R^{b] J_1 J_2}, \quad [\hat{K}^{ab}, \tilde{R}_c^{J_1J_2}] = -\delta_c^{[a} R^{b] J_1 J_2}$$

and the level minus two and one generators

$$\begin{aligned} [\tilde{R}_{ab}^I J, R^{cK_1K_2}] &= -4\delta_{[a}^c \delta_{J]}^{[K_1} \tilde{R}_{b]}^{I|K_2]} + \frac{1}{2} \delta_{[a}^c \delta_J^I \tilde{R}_{b]}^{K_1K_2} \\ [\tilde{R}_{ab}^I J, R^c_{K_1K_2}] &= 4\delta_{[a}^c \delta_{K_1]}^I \tilde{R}_{b]}^{J|K_2]} - \frac{1}{2} \delta_{[a}^c \delta_J^I \tilde{R}_{b]}^{K_1K_2} \\ [\tilde{R}_{ab}^{I_1 \dots I_4}, R^c_{K_1K_2}] &= 2\delta_{[a}^c \delta_{K_1K_2}^{[I_1I_2} \tilde{R}_{b]}^{I_3I_4]} \\ [\tilde{R}_{ab}^{I_1 \dots I_4}, R^{cK_1K_2}] &= \frac{2}{4!} \epsilon^{I_1 \dots I_4 K_1 K_2 J_1 J_2} \delta_{[a}^c \tilde{R}_{b]}^{J_1 J_2} \\ [\tilde{\hat{K}}_{ab}, R^c_{J_1J_2}] &= -\delta_{[a}^c \tilde{R}_{b]}^{J_1 J_2}, \quad [\tilde{\hat{K}}_{ab}, R^{cJ_1J_2}] = -\delta_{[a}^c \tilde{R}_{b]}^{J_1 J_2} \end{aligned} \quad (2.2.18)$$

The identification with the level two generators given in equation (2.2.8) in the eleven dimensional formulation of E_{11} is given by

$$\begin{aligned} R^{a_1 a_2 i}_8 &= \dot{R}^{a_1 a_2 i}, \quad R^{a_1 a_2 i}_j = \frac{2}{6!} \epsilon_{j l_1 \dots l_6} \dot{R}^{a_1 a_2 l_1 \dots l_6, i} - \frac{\delta_j^i}{2.6!} \epsilon_{l_1 \dots l_7} \dot{R}^{a_1 a_2 l_1 \dots l_6, l_7} \\ R^{a_1 a_2 8}_j &= \frac{d_1}{7!} \epsilon_{l_1 \dots l_7} \dot{R}^{a_1 a_2 l_1 \dots l_7, j}, \quad R^{a_1 a_2 i_1 \dots i_4} = -\frac{1}{2.3} \dot{R}^{a_1 a_2 i_1 \dots i_4}, \\ R^{a_1 a_2 i_1 i_2 i_3 8} &= -\frac{d_2}{7!} \epsilon_{l_1 \dots l_7} \dot{R}^{a_1 a_2 l_1 \dots l_7, i_1 i_2 i_3}, \quad \hat{K}^{ab} = \frac{2}{7} \epsilon_{l_1 \dots l_7} \dot{R}^{(a|l_1 \dots l_7, |b)} \end{aligned} \quad (2.2.19)$$

The constants d_1 and d_2 are yet to be fixed but they are not needed for the derivation of the above commutators. We note that $R^{a_1 a_2 8}_8 = -\sum_k R^{a_1 a_2 k}_k$

We now write the l_1 representation in terms of $SL(8)$ representations using similar techniques. In terms of these representations equation (2.12) can be written as

$$P_a(4, 1); Z^{IJ}(1, 28), Z_{IJ}(1, \bar{28}), \dots \quad (2.2.20)$$

The identification with the $SL(7)$ representations of equation (2.1.7) is given by

$$\begin{aligned} Z^{ij} &= \dot{Z}^{ij}, \quad Z^{j8} = \frac{1}{3.7!} \epsilon_{i_1 \dots i_7} \dot{Z}^{i_1 \dots i_7, j}, \\ Z_{i_1 i_2} &= \frac{1}{5!} \epsilon_{i_1 i_2 j_1 \dots j_5} \dot{Z}^{j_1 \dots j_5}, \quad Z_{i8} = P_i \end{aligned} \quad (2.2.21)$$

Using these identifications and the commutators of appendix A we find the commutation relations with the generators of E_7 are given by

$$[R^{I_1 \dots I_4}, Z^{J_1 J_2}] = \frac{1}{4!} \epsilon^{I_1 \dots I_4 J_1 J_2 K_1 K_2} Z_{K_1 K_2}, \quad [R^{I_1 \dots I_4}, Z_{J_1 J_2}] = \delta_{J_1 J_2}^{[I_1 I_2} Z^{I_3 I_4]}$$

$$\begin{aligned}
[K^I{}_J, Z^{L_1 L_2}] &= \delta_J^{L_1} Z^{I L_2} - \delta_J^{L_2} Z^{I L_1} - \frac{1}{4} \delta_J^I Z^{L_1 L_2}, \quad [K^I{}_J, P_a] = 0 \\
[K^I{}_J, Z_{L_1 L_2}] &= -\delta_{L_1}^I Z_{J L_2} + \delta_{L_2}^I Z_{J L_1} + \frac{1}{4} \delta_J^I Z_{L_1 L_2}
\end{aligned} \tag{2.2.22}$$

Their commutators with the level one E_{11} generators are given by

$$[R^{a I_1 I_2}, P_b] = \delta_b^a Z^{I_1 I_2}, \quad [R^a{}_{I_1 I_2}, P_b] = \delta_b^a Z_{I_1 I_2} \tag{2.2.23}$$

and with those at level minus one by

$$\begin{aligned}
[\tilde{R}_{a I_1 I_2}, Z^{J_1 J_2}] &= 2\delta_{J_1 J_2}^{I_1 I_2} P_a, \quad [\tilde{R}_{a I_1 I_2}, Z_{J_1 J_2}] = 0, \quad [\tilde{R}_{a I_1 I_2}, P_a] = 0 \\
[\tilde{R}_a{}^{I_1 I_2}, Z^{J_1 J_2}] &= 0, \quad [\tilde{R}_a{}^{I_1 I_2}, Z_{J_1 J_2}] = -2\delta_{I_1 I_2}^{J_1 J_2} P_a, \quad [\tilde{R}_a{}^{I_1 I_2}, P_a] = 0
\end{aligned} \tag{2.2.24}$$

2.3 The Cartan involution invariant subalgebra of E_{11} in four dimensions

In the last section we found the E_{11} algebra and the l_1 representations in terms of $GL(4) \otimes SL(8)$ representations. We now use this result to find the Cartan involution invariant subalgebra of E_{11} denoted $I_c(E_{11})$. A discussion of the Cartan involution can be found in earlier papers on E_{11} or in the book [17]. The first step is to find the Cartan involutions invariant subalgebra of E_7 . In terms of the $SL(8)$ formulation of the E_{11} algebra of equation (2.2.8), the Cartan involution, denoted by I_c ($I_c^2 = I$), acts on the level zero generators as

$$I_c(K^I{}_J) = -K^J{}_I, \quad I_c(R^{I_1 \dots I_4}) = -\star R^{I_1 \dots I_4} \equiv -\frac{1}{4!} \epsilon^{I_1 \dots I_4 J_1 \dots J_4} R^{J_1 \dots J_4}, \tag{2.3.1}$$

on the level one generators as

$$I_c(R^{a I_1 I_2}) = -\tilde{R}_{a I_1 I_2}, \quad I_c(R^a{}_{I_1 I_2}) = \tilde{R}_a{}^{I_1 I_2} \tag{2.3.2}$$

and on the level two generators as

$$I_c(R^{a_1 a_2 I}{}_J) = -\tilde{R}_{a_1 a_2}{}^J{}_I, \quad I_c(R^{a_1 a_2 I_1 \dots I_4}) = \star \tilde{R}_{a_1 a_2}{}^{I_1 \dots I_4}, \quad I_c(\hat{K}^{ab}) = -\tilde{\hat{K}}^{ab} \tag{2.3.3}$$

One way to find these results is to use the identifications with the generators of the eleven dimensional theory, given in the previous section, and the known action of the Cartan involution on the generators. In particular, we have that

$$\begin{aligned}
I_c(K^{\hat{a}}{}_{\hat{b}}) &= -K^{\hat{b}}{}_{\hat{a}}, \quad I_c(R^{\hat{a}_1 \hat{a}_2 \hat{a}_3}) = -R_{\hat{a}_1 \hat{a}_2 \hat{a}_3}, \\
I_c(R^{\hat{a}_1 \dots \hat{a}_6}) &= R_{\hat{a}_1 \dots \hat{a}_6}, \quad I_c(R^{\hat{a}_1 \dots \hat{a}_8, \hat{b}}) = -R_{\hat{a}_1 \dots \hat{a}_8, \hat{b}}
\end{aligned}$$

The positive sign in the third equation may seem incongruous, but it depends how one defines the generator $R_{\hat{a}_1 \dots \hat{a}_6}$ and the original papers made an unfortunate choice that we are now stuck with. For example, using equation (2.2.11) we find that

$$I_c(R^{a i_1 i_2}) = I_c(\dot{R}^{a i_1 i_2}) = -\dot{R}_{a i_1 i_2} = -\tilde{R}_{a i_1 i_2} \tag{2.3.4}$$

or, using equation (2.2.4), we find that

$$I_c(R^{i_1 i_2 i_3 8}) = I_c(\frac{1}{12}\dot{R}^{i_1 i_2 i_3}) = -\frac{1}{12}\dot{R}_{i_1 i_2 i_3} = -\frac{1}{4!}\epsilon_{i_1 i_2 i_3 j_1 \dots j_4} R^{j_1 j_2 j_3 j_4} = -\star R^{i_1 i_2 i_3 8}$$

At level zero the Cartan involution invariant subalgebra contains the generators

$$J^I{}_J = K^I{}_J - K^J{}_I, \quad \text{and} \quad S^{I_1 \dots I_4} = R^{I_1 \dots I_4} - \star R^{I_1 \dots I_4} \quad (2.3.5)$$

as well as the four dimensional Lorentz generators

$$J^a{}_b = K^a{}_b - K^b{}_a \quad (2.3.6)$$

We note that $\star S^{I_1 \dots I_4} = -S^{I_1 \dots I_4}$. The generators of equation (2.3.5) obey the algebra

$$[S^{I_1 \dots I_4}, S_{J_1 \dots J_4}] = -6\delta_{[J_1 \dots J_3}^{[I_1 \dots I_3} J^{I_4]}_{J_4]} + \frac{1}{4}\epsilon^{I_1 \dots I_4 L [J_1 J_2 J_3 J^{J_4]}_L} \quad (2.3.7)$$

as well as the commutation relations with $J^I{}_J$, which generate $\text{SO}(8)$, and act on other generators in the expected way, for example

$$[J^I{}_J, S^{K_1 \dots K_4}] = 4\delta_J^{[K_1} R^{I|K_2 K_3 K_4]} - 4\delta_I^{[K_1} R^{J|K_2 K_3 K_4]} \quad (2.3.8)$$

These commutators of equation (2.3.7) and (2.3.8) are those of $\text{SU}(8)$. This is to be expected as $\text{SU}(8)$ is well known to be the Cartan involution invariant subalgebra of E_7 .

At level plus and minus one the invariant generators are given by

$$S^{a I_1 I_2} = R^{a I_1 I_2} - \tilde{R}_{a I_1 I_2}, \quad \text{and} \quad \hat{S}^a{}_{I_1 I_2} = R^a{}_{I_1 I_2} + \tilde{R}_a{}^{I_1 I_2} \quad (2.3.9)$$

These 56 generators must transform as the $28 + \bar{28}$ -dimensional representations of $\text{SU}(8)$. However, the above generators mix under the $\text{SU}(8)$ commutators and the combinations that transform independently, that is, irreducibly, are given by

$$S_{\pm}^{a I_1 I_2} = S^{a I_1 I_2} \pm i\hat{S}^a{}_{I_1 I_2} \quad (2.3.10)$$

Indeed we find that

$$[S^{I_1 \dots I_4}, S_{\pm}^{a J_1 J_2}] = \mp \frac{i}{4!} \epsilon^{I_1 \dots I_4 J_1 J_2 K_1 K_2} S_{\pm}^a{}_{K_1 K_2} \pm i\delta_{J_1 J_2}^{[I_1 I_2} S_{\pm}^a{}_{I_3 I_4]} \quad (2.3.11)$$

The Cartan involution invariant generators at level plus and minus two are

$$S^{a_1 a_2 I}{}_J = R^{a_1 a_2 I}{}_J - \tilde{R}_{a_1 a_2}{}^J{}_I, \quad S^{a_1 a_2 I_1 \dots I_4} = R^{a_1 a_2 I_1 \dots I_4} + \star \tilde{R}_{a_1 a_2}{}^{I_1 \dots I_4}, \quad S^{ab} = \hat{K}^{ab} - \tilde{K}_{ab} \quad (2.3.12)$$

We note that $S^{a_1 a_2 K}{}_K = 0$, but otherwise this object has no particular symmetry. Similarly, unlike $S^{I_1 \dots I_4}$, which is anti-self dual $S^{a_1 a_2 I_1 \dots I_4}$ is neither self-dual or anti-self dual.

Their commutators with the SO(8) generators is as one expects and those with the remaining SU(8) generators are given by

$$\begin{aligned}
[S^{I_1 \dots I_4}, S_S^{a_1 a_2 J}_K] &= -2\delta_K^{[I_1|} S_+^{a_1 a_2 J|I_2 I_3 I_4]} - 2\delta_J^{[I_1|} S_+^{a_1 a_2 K|I_2 I_3 I_4]} + \frac{1}{2}\delta_K^J S_+^{a_1 a_2 I_1 I_2 I_3 I_4]} \\
[S^{I_1 \dots I_4}, S_A^{a_1 a_2 J}_K] &= -2\delta_K^{[I_1|} S_-^{a_1 a_2 J|I_2 I_3 I_4]} + 2\delta_J^{[I_1|} S_-^{a_1 a_2 K|I_2 I_3 I_4]} \\
[S^{I_1 \dots I_4}, S_+^{a_1 a_2 J_1 \dots J_4}] &= \frac{4}{3}\delta_{[I_1 I_2 I_3}^{[J_1 J_2 J_3|} S_S^{a_1 a_2|J_4]}_{I_4]} + \frac{1}{18}\epsilon^{I_1 \dots I_4 L [J_1 J_2 J_3|} S_S^{a_1 a_2|J_4]}_L \\
[S^{I_1 \dots I_4}, S_-^{a_1 a_2 J_1 \dots J_4}] &= \frac{4}{3}\delta_{[I_1 I_2 I_3}^{[J_1 J_2 J_3|} S_A^{a_1 a_2|J_4]}_{I_4]} + \frac{1}{18}\epsilon^{I_1 \dots I_4 L [J_1 J_2 J_3|} S_A^{a_1 a_2|J_4]}_L \quad (2.3.13)
\end{aligned}$$

where

$$\begin{aligned}
S_S^{abI}_J &= \frac{1}{2}(S^{abI}_J + S^{abJ}_I), \quad S_A^{abI}_J = \frac{1}{2}(S^{abI}_J - S^{abJ}_I), \\
S_{\pm}^{abI_1 I_2 I_3 I_4} &= S^{abI_1 I_2 I_3 I_4} \pm \star S^{abI_1 I_2 I_3 I_4} \quad (2.3.14)
\end{aligned}$$

We can interpret these equations as meaning that $S_+^{a_1 a_2 J_1 \dots J_4}$ and $S_S^{a_1 a_2 J}_K$ form the $35 + 35 = 70$ -dimensional representations of SU(8) while $S_-^{a_1 a_2 J_1 \dots J_4}$ and $S_A^{a_1 a_2 J}_K$ form the $35 + 28 = 63$ -dimensional representations of SU(8). We recall that these generators belong to the 133-dimensional representation of E_7 . In fact the last term on the right-hand side of the last commutator in equation (2.3.13) is zero as $L = J_4$ and the object is antisymmetric.

The commutators of the Cartan involution invariant generators of equation (2.3.10) must give those of equations (2.3.5) and (2.3.12) and using the commutators of section (2.2) we find that

$$\begin{aligned}
[S_{\pm}^{aI_1 I_2}, S_{\pm}^{bJ_1 J_2}] &= -12S_{\pm}^{abI_1 I_2 J_1 J_2} \pm 8i\delta_{[J_1}^{[I_1} S_S^{abI_2]}_{J_2]} \\
[S_{\pm}^{aI_1 I_2}, S_{\mp}^{bJ_1 J_2}] &= -12S_{\mp}^{abI_1 I_2 J_1 J_2} - 8\delta_a^b \delta_{[J_1}^{[I_1} J^{I_2]}_{J_2]} - 4\delta_{J_1 J_2}^{I_1 I_2} J^a_b \\
&\mp 8i\delta_{[J_1}^{[I_1} S_A^{abI_2]}_{J_2]} \mp 4i\delta_{J_1 J_2}^{I_1 I_2} S^{ab} \pm 24i\delta_a^b S^{I_1 I_2 J_1 J_2} \quad (2.3.15)
\end{aligned}$$

While the commutators of the generators of equation (2.3.10) and those of equation (2.3.12), but keeping only those generators of levels plus and minus one, are given by

$$\begin{aligned}
[S_{\pm}^{aI_1 I_2}, S_S^{b_1 b_2 J}_K] &= \mp 4i\delta_a^{[b_1} \delta_{[I_1}^{(K|} S_{\mp}^{b_2|J]}_{|J)|I_2]} \pm \frac{i}{2}\delta_K^J \delta_a^{[b_1} S_{\mp}^{b_2]}_{I_1 I_2}, \\
[S_{\pm}^{aI_1 I_2}, S_A^{b_1 b_2 J}_K] &= \pm 4i\delta_a^{[b_1} \delta_{[I_1}^{[J} S_{\pm}^{b_2|J]}_{|K]|I_2]}, \quad (2.3.16) \\
[S_{\pm}^{aI_1 I_2}, S_{\pm}^{b_1 b_2 J_1 \dots J_4}] &= 2\delta_a^{[b_1} \{\delta_{I_1 I_2}^{[J_1 J_2|} S_{\mp}^{b_2|J_3 J_4]} + \frac{1}{4!}\epsilon^{J_1 \dots J_4 I_1 I_2 K_1 K_2} S_{\mp}^{b_2]}_{K_1 K_2}\} \\
[S_{\pm}^{aI_1 I_2}, T_{\pm}^{b_1 b_2 J_1 \dots J_4}] &= 2\delta_a^{[b_1} \{\delta_{I_1 I_2}^{[J_1 J_2|} S_{\pm}^{b_2|J_3 J_4]} - \frac{1}{4!}\epsilon^{J_1 \dots J_4 I_1 I_2 K_1 K_2} S_{\pm}^{b_2]}_{K_1 K_2}\} \\
[S_{\pm}^{aI_1 I_2}, \hat{T}^{b_1 b_2}] &= \pm i\delta_a^{[b_1} T_{\pm}^{b_2]I_1 I_2} \quad (2.3.17)
\end{aligned}$$

The full commutators would contain in addition generators at level plus and minus three which are beyond the level we are keeping.

Finally we give the transformations of the l_1 representation under the Cartan involution invariant subalgebra. Rather than work with $Z^{I_1 I_2}$ and $Z_{I_1 I_2}$ we will work with the irreducible representations of $SU(8)$ which are given by

$$X_{\pm}^{I_1 I_2} = Z^{I_1 I_2} \pm i Z_{I_1 I_2} \quad (2.3.18)$$

Using equation (2.2.22) we find that their commutators with the generators of $SU(8)$ are given by

$$[S^{I_1 \dots I_4}, X_{\pm}^{J_1 J_2}] = \pm i \delta_{[J_1 J_2]}^{[I_1 I_2]} X_{\pm}^{I_3 I_4}] \mp \frac{1}{4!} \epsilon^{I_1 \dots I_4 J_1 J_2 K_1 K_2} X_{\pm}^{K_1 K_2} \quad (2.3.19)$$

that is, as the $28 + \bar{28}$ of $SU(8)$ should.

Using equations (2.2.23) and (2.2.24) we also find that

$$[S_{\pm}^{a I_1 I_2}, P_b] = \delta_b^a X_{\pm}^{I_1 I_2}, \quad [S_{\pm}^{a I_1 I_2}, X_{\pm}^{a J_1 J_2}] = 0, \quad [S_{\pm}^{a I_1 I_2}, X_{\mp}^{a J_1 J_2}] = -4 \delta_{J_1 J_2}^{I_1 I_2} P_a \quad (2.3.20)$$

2.4 The decomposition of E_{11} into representations of its Cartan invariant subalgebra

The generators of the E_{11} algebra can be split into those that are invariant under the Cartan involution and those that transform with a minus sign. The former are those in the Cartan involution invariant algebra $I_c(E_{11})$ given in the previous section. The latter are sometimes called the coset generators and are the subject of this section. Clearly, the commutator of two elements of $I_c(E_{11})$ gives a result in $I_c(E_{11})$. Furthermore if $S \in I_c(E_{11})$ and T is a coset generator, that is $I_c(T) = -T$, then their commutator $[S, T]$ is also a coset generator since $I_c([S, T]) = [I_c(S), I_c(T)] = -[S, T]$. As a result, the coset generators belong to a representations of $I_c(E_{11})$. In this section we find the commutators of the coset generators with those of $I_c(E_{11})$. Put another way we wish to decompose the adjoint representation of E_{11} into representations of $I_c(E_{11})$. In the next section we will use these commutation relations to deduce the crucial field variations under which the theory must be invariant.

Using equation (2.3.1) we find that the level zero the coset generators are given by

$$T^{I_1 \dots I_4} = R^{I_1 \dots I_4} + \star R^{I_1 \dots I_4}, \quad T^{I_1}_{I_2} = K^{I_1}_{I_2} + K^{I_2}_{I_1}, \quad (2.4.1)$$

We note that $\sum_L T^L_L = 0$ and $\star T^{I_1 \dots I_4} = T^{I_1 \dots I_4}$. Their commutators with the generators of $SO(8)$ are obvious and those with the remaining generators of $SU(8)$ are given by

$$\begin{aligned} [S^{I_1 \dots I_4}, T^J_K] &= -4 \{ \delta_K^{[I_1} T^{J|I_2 \dots I_4]} - \frac{1}{4!} \epsilon_{I_1 \dots I_4 K L_1 L_2 L_3} T^{J L_1 L_2 L_3} \} + (K \leftrightarrow J) \\ &= \{ -8 \delta_K^{[I_1} T^{J|I_2 \dots I_4]} + \delta_K^J T^{I_1 I_2 \dots I_4} \} + (K \leftrightarrow J) \end{aligned} \quad (2.4.2)$$

and

$$[S^{I_1 \dots I_4}, T^{J_1 \dots J_4}] = 6\delta_{[J_1 J_2 J_3}^{[I_1 I_2 I_3} T^{I_4]}_{J_4]} + \frac{1}{4}\epsilon^{I_1 \dots I_4 L [J_1 J_2 J_3} T^{J_4]}_L \quad (2.4.3)$$

The generators of equation (2.4.1) belong to the 70-dimensional representations of SU(8).

The coset generators formed from the level one and minus one generators are given by

$$T^{a I_1 I_2} = R^{a I_1 I_2} + \tilde{R}_{a I_1 I_2}, \quad \hat{T}_{I_1 I_2}^a = R^a_{I_1 I_2} - \tilde{R}_a^{I_1 I_2} \quad (2.4.4)$$

As for the analogous objects in the Cartan involution invariant subalgebra, these two objects transform into each other under SU(8) and so we define instead the generators

$$T_{\pm}^{a I_1 I_2} = T^{a I_1 I_2} \pm i\hat{T}_{I_1 I_2}^a \quad (2.4.5)$$

Their commutators with the generators of SU(8) are given by

$$[S^{I_1 \dots I_4}, T_{\pm}^{a J_1 J_2}] = \mp \frac{i}{4!} \epsilon^{I_1 \dots I_4 J_1 J_2 K_1 K_2} T_{\pm K_1 K_2}^a \pm i\delta_{[I_1 I_2}^{J_1 J_2} T_{\pm I_3 I_4]}^a \quad (2.4.6)$$

and we recognise that they belong to the $28 + \bar{28}$ representations of SU(8). We note that the SU(8) representations do not emerge in the familiar form, for example the 63-dimensional representation is usually carried by a traceless object with one up and one down index whose ranges are one to eight; the count being $8 \cdot 8 - 1 = 63$. However, in the formulation we have in this paper it is instead carried by J^I_J and $S^{I_1 \dots I_4}$; the count being $\frac{8 \cdot 7}{2} + \frac{8 \cdot 7 \cdot 6 \cdot 5}{2 \cdot 4!} = 63$.

In previous papers we have labeled representations of SL(n) by giving the Dynkin diagram with the $n - 1$ dots in a horizontal row labeled from 1 to $n - 1$ from left to right and then taking the fundamental representation associated with node 1 to be the \bar{n} representation and carried by the tensor T_i , $i = 1, 2, \dots, n$ while the fundamental representation associated with node $n - 1$ to be the n representation and carried by the tensor T^i , $i = 1, 2, \dots, n$. The fundamental representation associated with node $n - 2$ is then the $n(n - 1)$ representation carried by the tensor $T^{i_1 i_2} = T^{[i_1 i_2]}$, etc. The use of over bars also applies to the SL(n) Dynkin diagram when embedded into the E_{n+1} Dynkin diagram, for example the fundamental representation associated with node one of the E_7 Dynkin diagram is the $\bar{27}$. However, given the unfamiliar way the SU(8) representations occur here we take, by definition, $T_+^{a I_1 I_2}$ to be the 28-dimensional representation of SU(8).

The coset generators formed from the level two and minus two generators are given by

$$T^{a_1 a_2 I}_J = R^{a_1 a_2 I}_{J+} + \tilde{R}_{a_1 a_2}^J{}_I, \quad T^{a_1 a_2 I_1 \dots I_4} = R^{a_1 a_2 I_1 \dots I_4} - \star \tilde{R}_{a_1 a_2}^{I_1 \dots I_4}, \quad \hat{T}^{ab} = \hat{K}^{ab} + \tilde{K}_{ab} \quad (2.4.7)$$

We note that $T^{a_1 a_2 K}_K = 0$, but otherwise this object has no particular symmetry. Similarly, $T^{a_1 a_2 I_1 \dots I_4}$ is neither self-dual or anti-self dual. Their commutators with the SO(8) generators is as one expects and those with the remaining SU(8) generators are given by

$$[S^{I_1 \dots I_4}, T_S^{a_1 a_2 J}_K] = -2\delta_K^{[I_1} T_+^{a_1 a_2 J | I_2 I_3 I_4]} - 2\delta_J^{[I_1} T_+^{a_1 a_2 K | I_2 I_3 I_4]} + \frac{1}{2}\delta_K^J T_+^{a_1 a_2 I_1 I_2 I_3 I_4]}$$

$$\begin{aligned}
[S^{I_1 \dots I_4}, T_A^{a_1 a_2 J}{}_K] &= -2\delta_K^{[I_1} T_-^{a_1 a_2 J|I_2 I_3 I_4]} + 2\delta_J^{[I_1} T_-^{a_1 a_2 K|I_2 I_3 I_4]} \\
[S^{I_1 \dots I_4}, T_+^{a_1 a_2 J_1 \dots J_4}] &= \frac{4}{3}\delta_{[I_1 I_2 I_3}^{[J_1 J_2 J_3|} T_S^{a_1 a_2 |J_4]}_{I_4]} + \frac{1}{18}\epsilon^{I_1 \dots I_4 L [J_1 J_2 J_3|} T_S^{a_1 a_2 |J_4]}_L \\
[S^{I_1 \dots I_4}, T_-^{a_1 a_2 J_1 \dots J_4}] &= \frac{4}{3}\delta_{[I_1 I_2 I_3}^{[J_1 J_2 J_3|} T_A^{a_1 a_2 |J_4]}_{I_4]} + \frac{1}{18}\epsilon^{I_1 \dots I_4 L [J_1 J_2 J_3|} T_A^{a_1 a_2 |J_4]}_L
\end{aligned} \tag{2.4.8}$$

where

$$\begin{aligned}
T_S^{abI}{}_J &= \frac{1}{2}(T^{abI}{}_J + T^{abJ}{}_I), \quad T_A^{abI}{}_J = \frac{1}{2}(T^{abI}{}_J - T^{abJ}{}_I), \\
T_{\pm}^{abI_1 I_2 I_3 I_4} &= T^{abI_1 I_2 I_3 I_4} \pm \star T^{abI_1 I_2 I_3 I_4}
\end{aligned} \tag{2.4.9}$$

We can interpret these equations as meaning that $T_+^{a_1 a_2 J_1 \dots J_4}$ and $T_S^{a_1 a_2 J}{}_K$ form the $35 + 35 = 70$ -dimensional representations of $SU(8)$ while $T_-^{a_1 a_2 J_1 \dots J_4}$ and $T_A^{a_1 a_2 J}{}_K$ form the $35 + 28 = 63$ -dimensional representations of $SU(8)$. We recall that these generators belong to the 133-dimensional representation of E_7 .

The Cartan involution invariant subalgebra $I_c(E_{11})$ can be constructed from the multiple commutators of the generators of $SU(8)$ and the generators $S_{\pm}^{a I_1 I_2}$ of equation (2.3.10). As such to know the commutators of the coset generators with all of those of the Cartan involution subalgebra it suffices to find the commutators of the coset generators with the generators $J^I{}_K, S^{I_1 \dots I_4}$ of $SU(8)$ and $S_{\pm}^{a I_1 I_2}$. The former were given above and we now give the commutators of the latter; the result with those of level zero of equation (2.4.1) are given by

$$\begin{aligned}
[S_{\pm}^{a I_1 I_2}, T^J{}_K] &= -2\delta_J^{[I_1} T_{\mp}^{a |K|I_2]} - 2\delta_K^{[I_1} T_{\mp}^{a |J|I_2]}, \quad [S_{\pm}^{a I_1 I_2}, T^{b_1 b_2}] = -2\delta^{a(b_1} T_{\pm}^{b_2) I_1 I_2}, \\
[S_{\pm}^{a I_1 I_2}, T^{J_1 \dots J_4}] &= \mp i \delta_{I_1 I_2}^{[J_1 J_2|} T_{\mp}^{a |J_3 J_4]} \mp \frac{i}{4!} \epsilon^{J_1 \dots J_4 I_1 I_2 K_1 K_2} T_{\mp}^{a}{}_{K_1 K_2}
\end{aligned} \tag{2.4.10}$$

with the coset generators of equation (2.4.4) by

$$\begin{aligned}
[S_{\pm}^{a I_1 I_2}, T_{\pm}^{b J_1 J_2}] &= 12T_{\pm}^{ab}{}_{I_1 I_2 J_1 J_2} + 8\delta_b^{[I_1} \delta_{[J_1}^{I_2} T_{\pm}^{a |J_2]}_{J_2]} \pm 24i\delta_b^a T^{I_1 I_2 J_1 J_2} \mp 8i\delta_{[J_1}^{[I_1} T_S^{ab I_2]}_{J_2]} \\
[S_{\pm}^{a I_1 I_2}, T_{\mp}^{b J_1 J_2}] &= 12T_{\mp}^{ab I_1 I_2 J_1 J_2} + \delta_{J_1 J_2}^{I_1 I_2} (4T^{ab} - 2 \sum_c T^c{}_c) \pm 8i\delta_{[J_1}^{[I_1} T_A^{ab I_2]}_{J_2]} \mp 4i\delta_{J_1 J_2}^{I_1 I_2} \hat{T}^{ab}
\end{aligned} \tag{2.4.11}$$

While the commutators of the generators of equation (2.4.9) with $S_{\pm}^{a I_1 I_2}$, but keeping only those generators of levels plus and minus one, are given by

$$\begin{aligned}
[S_{\pm}^{a I_1 I_2}, T_S^{b_1 b_2 J}{}_K] &= \mp 4i\delta_a^{[b_1} \delta_{[I_1}^{(K} T_{\mp}^{b_2 |J|b_2]}_{|J|I_2]} \pm \frac{i}{2} \delta_K^J \delta_a^{[b_1} T_{\mp}^{b_2]}_{I_1 I_2]}, \\
[S_{\pm}^{a I_1 I_2}, T_A^{b_1 b_2 J}{}_K] &= \pm 4i\delta_a^{[b_1} \delta_{[I_1}^{[J} T_{\pm}^{b_2 |J|b_2]}_{|K|I_2]}], \\
[S_{\pm}^{a I_1 I_2}, T_{\mp}^{b_1 b_2 J_1 \dots J_4}] &= 2\delta_a^{[b_1} \{\delta_{I_1 I_2}^{[J_1 J_2|} T_{\mp}^{b_2 |J_3 J_4]} + \frac{1}{4!} \epsilon^{J_1 \dots J_4 I_1 I_2 K_1 K_2} T_{\mp}^{b_2]}_{K_1 K_2}\}
\end{aligned} \tag{2.4.12}$$

$$[S_{\pm}^{aI_1I_2}, T_{\pm}^{b_1b_2J_1\dots J_4}] = 2\delta_a^{[b_1}\{\delta_{I_1I_2}^{J_1J_2}|T_{\pm}^{b_2]|J_3J_4}] - \frac{1}{4!}\epsilon^{J_1\dots J_4I_1I_2K_1K_2}T_{\pm}^{[b_2]}{}_{K_1K_2}\} \\ [S_{\pm}^{aI_1I_2}, \hat{T}^{b_1b_2}] = \pm i\delta^{a(b_1}T_{\pm}^{b_2)I_1I_2} \quad (2.4.13)$$

Finally we give the transformations of the l_1 representation under the Cartan involution invariant subalgebra. Rather than work with $Z^{I_1I_2}$ and $Z_{I_1I_2}$ we will work with the irreducible representations of $SU(8)$ which are given by

$$X_{\pm}^{I_1I_2} = Z^{I_1I_2} \pm iZ_{I_1I_2} \quad (2.4.14)$$

Using equation (2.2.22) we find that their commutators with the generators of $SU(8)$ are given by

$$[S^{I_1\dots I_4}, X_{\pm}^{J_1J_2}] = \pm i\delta_{[J_1J_2]}^{[I_1I_2]}X_{\pm}^{I_3I_4}] \mp \frac{1}{4!}\epsilon^{I_1\dots I_4J_1J_2K_1K_2}X_{\pm}^{K_1K_2} \quad (2.4.15)$$

that is, as the $28 + \bar{28}$ of $SU(8)$ should.

Using equations (2.2.23) and (2.2.24) we also find that

$$[S_{\pm}^{aI_1I_2}, P_b] = \delta_b^a X_{\pm}^{I_1I_2}, \quad [S_{\pm}^{aI_1I_2}, X_{\pm}^{aJ_1J_2}] = 0, \quad [S_{\pm}^{aI_1I_2}, X_{\mp}^{aJ_1J_2}] = -4\delta_{J_1J_2}^{I_1I_2}P_a \quad (2.4.16)$$

3 The Cartan forms and generalised vielbein

We can finally construct the building blocks of the non-linear realisation of $E_{11} \otimes_s l_1$ appropriate to four dimensions, meaning the semi-direct product algebra constructed from E_{11} and its l_1 representations l_1 . In this construction the commutators of the generators of E_{11} with themselves are just those of E_{11} . The commutators of generators of E_{11} with those in the l_1 representations result in generators in the l_1 representation and the Jacobi identities then imply that the structure constants are just the matrices of the l_1 representation. Clearly one can carry out this construction for any group and one of its representations. Physicists are very familiar with semi-direct product algebras as the Poincaré algebra is the semi-direct product of the translations and the Lorentz group. We take the generators of the l_1 to commute, but more sophisticated commutators are possible.

We begin with a generic group element $g \in E_{11} \otimes_s l_1$ which can be written as

$$g = g_l g_E \quad (3.1)$$

where

$$g_E = g_{-1}g_{-2}\dots g_0\dots g_2g_1 \quad (3.2)$$

where g_n contains level n generators; those with positive level are given by

$$g_0 = e^{h_a{}^b K^a{}_b} e^{\phi^I{}_J R^J{}_I} e^{\phi_{I_1\dots I_4} R^{I_1\dots I_4}} \equiv g_h g_{\phi}, \quad g_1 = e^{A_{aI_1I_2} R^{aI_1I_2} + A_a{}^{I_1I_2} R^a{}_{I_1I_2}}, \\ g_2 = e^{\hat{h}_{ab} \hat{K}^{ab}} e^{A_{a_1a_2}{}^I{}_J R^{a_1a_2}{}^J{}_I} e^{A_{a_1a_2I_1\dots I_4} R^{a_1a_2I_1\dots I_4}} \quad (3.3)$$

In this and the next equation we have used the generators in their $SL(8)$ basis. The group element formed from the generators of the l_1 representations is given by

$$g_l = e^{x^a P_a} e^{x_{I_1 I_2} Z^{I_1 I_2}} e^{x^{I_1 I_2} Z_{I_1 I_2}} e^{\hat{x}_a Z^a} e^{x_a{}^I{}_J Z^a{}^J{}_I} e^{x_{a I_1 \dots I_4} Z^a{}^{I_1 \dots I_4}} \dots = e^{z^A L_A} \quad (3.4)$$

where we have denoted the generalised coordinates by z^A and the generators of the l_1 representation by L_A . Thus the non-linear realisation introduces a generalised space-time with the coordinates

$$x^a, x_{I_1 I_2}, x^{I_1 I_2}, \hat{x}_a, x_a{}^I{}_J, x_{a I_1 \dots I_4}, \dots \quad (3.5)$$

The fields that occur in the group element g_E are taken to depend on the generalised space-time that is the coordinates of equation (3.5).

The non-linear realisation is by definition just a set of dynamical equations, or Lagrangian, that is invariant under the transformations

$$g \rightarrow g_0 g, \quad g_0 \in E_{11} \otimes_s l_1, \quad \text{as well as} \quad g \rightarrow g h, \quad h \in I_c(E_{11}) \quad (3.6)$$

The group element g_0 is a rigid transformation, that is a constant, while h is a local transformation, that is it depends on the generalised space-time. As the generators in g_l form a representation of E_{11} the above transformations for $g_0 \in E_{11}$ can be written as

$$g_l \rightarrow g_0 g_l g_0^{-1}, g_E \rightarrow g_0 g_E \quad \text{and} \quad g_E \rightarrow g_E h \quad (3.7)$$

As a consequence the coordinates are inert under the local transformations but transform under the rigid transformations as

$$z^A L_A \rightarrow g_0 z^A L_A g_0^{-1} = z^\Pi D(g_0^{-1})_\Pi{}^A L_A \quad (3.8)$$

Using the local transformation we may bring g_E into the form

$$g_E = g_0 \dots g_2 g_1 \quad (3.9)$$

Thus the theory contains the graviton field $h_a{}^b$, associated with the generators $K^a{}_b$ of $GL(4)$, the 70 scalars $\phi^I{}_J, \phi_{I_1 \dots I_4}$, associated with the generators $K^I{}_J$ and $R^{I_1 \dots I_4}$ respectively, as well as the gauge fields $A_{a I_1 I_2}$, $A_a{}^{I_1 I_2}$, $A_{a_1 a_2}{}^I{}_J$ and $A_{a_1 a_2 I_1 \dots I_4}$, associated with the level one and two generators, and in addition at level two we have the field \hat{h}_{ab} corresponding to the generator \hat{K}^{ab} which is the dual field of gravity [10]. The parameterisation of the group element differs from that used in some earlier works on E_{11} , but this does not affect any physical results.

As explained in the introduction, the l_1 representation contains all the brane charges and as it also leads to the generalised space-time there is a one to one relation between the brane charges and the coordinates of the generalised space-time. Furthermore, for every field in E_{11} there is a corresponding element in the l_1 representation. As such for every field there is an associated coordinate in the generalised space-time and an associated brane. For example, the metric $h_a{}^b$ corresponds to the space-time translations, that is the charge P_a , which is carried by the point particle, or pp-wave, and has associated

coordinate x^a , the dual graviton \hat{h}_{ab} corresponds to the charge \hat{Z}^a , which is carried by the Taub-NUT solution, and has associated coordinate \hat{x}_a , the gauge fields $A_{aI_1I_2}$ and $A_a^{I_1I_2}$ corresponds to the brane charges $Z^{I_1I_2}$ and $Z_{I_1I_2}$, which are the sources for corresponding brane solutions, and the associated coordinates are $x_{I_1I_2}$ and $x^{I_1I_2}$.

The dynamics is usually constructed from the Cartan forms $\mathcal{V} = g^{-1}dg$ as these are obviously inert under the E_{11} rigid transformations of equation (3.5) and only transform under the local transformations as

$$\mathcal{V} \rightarrow h^{-1}\mathcal{V}h + h^{-1}dh \quad (3.10)$$

Hence if we use the Cartan forms, the problem of finding a set of field equations which are invariant under equation (3.6) reduces to finding a set that is invariant under the local subalgebra $I_c(E_{11})$, that is the local transformations also given in equation (3.6) and so equation (3.10).

The Cartan forms can be written as

$$\mathcal{V} = \mathcal{V}_E + \mathcal{V}_l \quad (3.11)$$

where

$$\mathcal{V}_E = g_E^{-1}dg_E \quad \text{and} \quad \mathcal{V}_l = g_l^{-1}(g_l^{-1}dg_l)g_E \quad (3.12)$$

The first part \mathcal{V}_E is just the Cartan form for E_{11} while \mathcal{V}_l is a sum of generators in the l_1 representation. Both \mathcal{V}_E and \mathcal{V}_l are invariant under rigid transformations and under local transformations they change as

$$\mathcal{V}_E \rightarrow h^{-1}\mathcal{V}_Eh + h^{-1}dh \quad \text{and} \quad \mathcal{V}_l \rightarrow h^{-1}\mathcal{V}_lh \quad (3.13)$$

Let us evaluate the E_{11} part of the Cartan form

$$\begin{aligned} \mathcal{V}_E = dz^\Pi G_{\Pi, \bullet} R^\bullet &= G_a{}^b K^a{}_b + \Omega^I{}_J K^J{}_I + \Omega_{I_1 \dots I_4} R^{I_1 \dots I_4} \\ &+ G_{aI_1I_2} R^{aI_1I_2} + G_a{}^{I_1I_2} R^a{}_{I_1I_2} \\ &+ \hat{G}_{ab} \hat{K}^{ab} + G_{a_1a_2}{}^I{}_J R^{a_1a_2J}{}_I + G_{a_1a_2I_1 \dots I_4} R^{a_1a_2I_1 \dots I_4} + \dots \end{aligned} \quad (3.14)$$

where \bullet denotes the indices on the generators of E_{11} . Explicitly one finds that

$$\begin{aligned} G_a{}^b &= (e^{-1}de)_a{}^b, \quad G_\phi = g_\phi^{-1}dg_\phi \equiv \Omega^J{}_I K^I{}_J + \Omega_{I_1 \dots I_4} R^{I_1 \dots I_4} \\ G_{aI_1I_2} &= \tilde{D}A_{aI_1I_2}, \quad G_a{}^{I_1I_2} = \tilde{D}A_a{}^{I_1I_2}, \\ G_{a_1a_2}{}^I{}_J &= \tilde{D}A_{a_1a_2}{}^I{}_J - 2A_{[a_1|LJ}\tilde{D}A_{|a_2]}{}^{LI} - 2A_{[a_1]}{}^{LI}\tilde{D}A_{|a_2]}{}_{LJ} \\ G_{a_1a_2I_1 \dots I_4} &= \tilde{D}A_{a_1a_2I_1 \dots I_4} + 6A_{[a_1|I_1I_2}\tilde{D}A_{|a_2]I_3I_4]} - \frac{1}{4}\epsilon_{I_1 \dots I_4J_1 \dots J_4}A_{[a_1]}{}^{J_1J_2}\tilde{D}A_{|a_2]}{}^{J_3J_4} \\ \hat{G}_{ab} &= \tilde{D}\hat{h}_{ab} - A_{(a_1|I_1I_2}\tilde{D}A_{|a_2)}{}^{I_2I_3} + A_{(a_1|}{}^{I_1I_2}\tilde{D}A_{|a_2)}{}_{I_2I_3} \end{aligned} \quad (3.15)$$

where $e_\mu^a \equiv (e^h)_\mu^a$ and

$$\begin{aligned}\tilde{D}A_{aI_1I_2} &\equiv dA_{aI_1I_2} + (e^{-1}de)_a^b A_{bI_1I_2} + 2\Omega^J_{[I_1|} A_{aJ|I_2]} + \Omega_{I_1\dots I_4} A_a^{I_3I_4}, \\ \tilde{D}A_a^{I_1I_2} &\equiv dA_a^{I_1I_2} + (e^{-1}de)_a^b A_b^{I_1I_2} - 2\Omega^{[I_1|}{}_J A_a^{J|I_2]} - \Omega_{I_1I_2J_1J_2} A_a^{J_1J_2}\end{aligned}\quad (3.16)$$

with analogous expressions for other quantities.

Let us now evaluate the part of the Cartan form in equation (3.11) containing the generators of the l_1 representation; we may write it as

$$\begin{aligned}\mathcal{V}_l &= g^{-1}dg = dz^\Pi E_\Pi^A L_A \\ &= g_E^{-1}(dx^a P_a + dx_{I_1I_2} Z^{I_1I_2} + dx^{I_1I_2} Z_{I_1I_2} + Z^a d\hat{x}_a + x_a^I{}_J Z^a{}_J{}_I + dx_{aI_1\dots I_4} Z^{aI_1\dots I_4} + \dots)g_E \\ &= E^a P_a + E_{I_1I_2} Z^{I_1I_2} + E^{I_1I_2} Z_{I_1I_2} + \dots\end{aligned}\quad (3.17)$$

where $E^A = dz^\Pi E_\Pi^A$. Using equations (2.2.22-24) we find that E_Π^A , viewed as a matrix, is given at low orders by

$$E = \begin{pmatrix} (dete)^{-\frac{1}{2}} e_\mu^a & -(dete)^{-\frac{1}{2}} e_\mu^c A_{cJ_1J_2} & -(dete)^{-\frac{1}{2}} e_\mu^c A_c^{J_1J_2} \\ 0 & \mathcal{N}^{I_1I_2}{}_{J_1J_2} & \mathcal{N}^{I_1I_2J_1J_2} \\ 0 & \mathcal{N}_{I_1I_2J_1J_2} & \mathcal{N}_{I_1I_2}{}^{J_1J_2} \end{pmatrix}\quad (3.18)$$

The matrix \mathcal{N} is the vielbein in the scalar sector, that is $g_\phi^{-1}(dx_{I_1I_2} Z^{I_1I_2} + dx^{I_1I_2} Z_{I_1I_2})g_\phi \equiv dx \cdot \mathcal{N} \cdot l$. This illustrates the fact that the non-linear realisation leads to a generalised space-time with a generalised tangent space, which for the four dimensional theory consists of the usual tangent space of four-dimensional space-time, a 56-dimensional tangent space and then higher level tangent spaces. The tangent space can be read off from the l_1 representation in an obvious way. The tangent space group is $I_c(E_{11})$. At lowest level in four dimensions the tangent space group is $SO(4) \otimes SU(8)$ and the tangent vectors transform, at lowest level, in the 4 representation of $SO(4)$ and the $28 + \bar{28}$ representations of $SU(8)$. It will prove advantageous to express the tangent space in terms of objects that transform into themselves, that is, identify precisely, the 28 and $\bar{28}$ of $SU(8)$. To this end we can rewrite \mathcal{V}_l at lowest order as

$$E^a P_a + E_{I_1I_2} Z^{I_1I_2} + E^{I_1I_2} Z_{I_1I_2} = E^a P_a + E_{+I_1I_2} X_+^{I_1I_2} + E_-^{I_1I_2} X_{-I_1I_2}\quad (3.19)$$

using the generators of equation (2.4.14). Comparing terms we find that

$$E_{\pm I_1I_2} = \frac{1}{2}(E_{I_1I_2} \mp iE^{I_1I_2})\quad (3.20)$$

As we see, the non-linear realisation $E_{11} \otimes_s l_1$ automatically encodes a generalised geometry equipped with a generalised vielbein which will be given explicitly at low levels shortly.

Our task is to find a set of dynamics which is invariant under the rigid and local transformations of equation (3.7) and with this in mind we now consider in more detail the transformations of the two parts of the Cartan form beginning with the E_{11} part,

\mathcal{V}_E . As noted above the Cartan forms only transform under local $I_c(E_{11})$ transformations. It is useful to introduce the operation $g^* = (I_c(g))^{-1}$ on the group. While I_c is an automorphism, i.e. on two group elements $I_c(g_1 g_2) = I_c(g_1) I_c(g_2)$, the action of $*$ reverses the order, that is $(g_1 g_2)^* = (g_2)^* (g_1)^*$. The action of $*$ on the algebra is given by $A^* = -I_c(A)$ and $(AB)^* = B^* A^*$. A group element belonging to $I_c(E_{11})$ obeys $h^* = h^{-1}$ and the two transformations of equation (3.7) imply that $g^* \rightarrow h^{-1} g^* (g_0)^*$. We write the Cartan forms \mathcal{V}_E as

$$\mathcal{V}_E = P + Q, \quad \text{where } P = \frac{1}{2}(\mathcal{V}_E + \mathcal{V}_E^*), \quad Q = \frac{1}{2}(\mathcal{V}_E - \mathcal{V}_E^*) \quad (3.21)$$

and then the transformations of equation (3.13) become

$$P \rightarrow h^{-1} P h, \quad Q \rightarrow h^{-1} Q h + h^{-1} d h \quad (3.22)$$

Examining equation (3.14) we find that

$$\begin{aligned} 2P &= G_a^b T_a^b + \Omega^J_I T^I_J + \Omega_{I_1 \dots I_4} T^{I_1 \dots I_4} + G_{a I_1 I_2} T^{a I_1 I_2} + G_a^{I_1 I_2} T^a_{I_1 I_2} \\ &\quad + \hat{G}_{ab} \hat{T}^{ab} + G_{a_1 a_2}^I J R^{a_1 a_2 J}_I + G_{a_1 a_2 I_1 \dots I_4} T^{a_1 a_2 I_1 \dots I_4} + \dots \\ &= G_a^b T_a^b + \Omega^J_I T^I_J + \Omega_{I_1 \dots I_4} T^{I_1 \dots I_4} + G_{a+I_1 I_2} T_+^{a I_1 I_2} + G_{a-I_1 I_2} T_-^{a I_1 I_2} \\ &\quad + \hat{G}_{ab} \hat{T}^{ab} + G_{a_1 a_2 S}^I J T_S^{a_1 a_2 J}_I + G_{a_1 a_2 A}^I J T_A^{a_1 a_2 J}_I \\ &\quad + G_{a_1 a_2 + I_1 \dots I_4} T_+^{a_1 a_2 I_1 \dots I_4} + G_{a_1 a_2 - I_1 \dots I_4} T_-^{a_1 a_2 I_1 \dots I_4} + \dots \end{aligned} \quad (3.23)$$

and

$$\begin{aligned} 2Q &= G_a^b J_a^b + \Omega^J_I J^I_J + \Omega_{I_1 \dots I_4} S^{I_1 \dots I_4} + G_{a I_1 I_2} S^{a I_1 I_2} + G_a^{I_1 I_2} S^a_{I_1 I_2} \\ &\quad + \hat{G}_{ab} \hat{S}^{ab} + G_{a_1 a_2}^I J S^{a_1 a_2 J}_I + G_{a_1 a_2 I_1 \dots I_4} S^{a_1 a_2 I_1 \dots I_4} + \dots \end{aligned} \quad (3.24)$$

Where in the first line we have used the generators of equations (2.4.1), (2.4.4) (2.4.7) and in the second line the generators of equations (2.4.5) and (2.4.9) which transform as irreducible representation under $SU(8)$. The Cartan forms inherit the properties of the generators from which they arise; for example $G_{a_1 a_2 + I_1 \dots I_4}$ and $G_{a_1 a_2 - I_1 \dots I_4}$ are self dual and anti-self dual. We note that except for the level zero generators the connection Q contains the same objects as the covariant quantity P .

Taking $h = 1 - \Lambda_{a+I_1 I_2} S_+^{a I_1 I_2} - \Lambda_{a-I_1 I_2} S_-^{a I_1 I_2}$, the local transformations of P of equation (3.23) implies, using the equations of section four that

$$\begin{aligned} \delta G^J_K &= \sum_{\pm} (4\Lambda_{a\pm L K} G_{a\pm L}^J + 4\Lambda_{a\pm L J} G_{a\pm L}^K - \delta_K^J \Lambda_{a\pm L M} G_{a\pm L}^M), \\ \delta G^{I_1 I_2 I_3 I_4} &= 12i \sum_{\pm} \pm (\Lambda_{a\pm [I_1 I_2]} G_{a\pm [I_3 I_4]} + \frac{1}{4!} \epsilon_{I_1 \dots I_4}^{J_1 \dots J_4} \Lambda_{a\pm J_1 J_2} G_{a\pm J_3 J_4}) \\ \delta G_{a_1 a_2} &= \sum_{\pm} (4\Lambda_{(a_1 | \pm I_1 I_2} G_{| a_2) \mp I_1 I_2} - 2\delta_{a_1 a_2} \Lambda_{b\pm I_1 I_2} G_{b\mp I_1 I_2}) \end{aligned}$$

$$\begin{aligned}
\delta G_{a \pm I_1 I_2} &= \pm 2i \Lambda_{a \mp J_1 J_2} G^{I_1 I_2 J_1 J_2} + 4 \Lambda_{a \mp J [I_1} G_{I_2]}^J + 4 \Lambda_{b \mp J_1 J_2} G^b_{a+J_1 J_2 I_1 I_2} \\
&\quad + 4 \Lambda_{b \pm J_1 J_2} G^b_{a-J_1 J_2 I_1 I_2} \pm 4i \Lambda_{b \mp J [I_1} G^b_{a S^J | I_2]} \mp 4i \Lambda_{b \pm J [I_1} G^b_{a A | I_2]}^J \\
&\quad + \Lambda_{b \pm I_1 I_2} (\pm i \hat{G}^b_a - 2 G^b_a) \\
\delta G_{a_1 a_2 - I_1 \dots I_4} &= 6 \sum_{\pm} (\Lambda_{[a_1 | \pm [I_1 I_2 |} G_{| a_2] | \mp [I_3 I_4]} - \frac{1}{4!} \epsilon_{I_1 \dots I_4}^{J_1 \dots J_4} \Lambda_{[a_1 | \pm J_1 J_2} G_{| a_2] | \mp J_3 J_4}), \\
\delta G_{a_1 a_2 + I_1 \dots I_4} &= 6 \sum_{\pm} (\Lambda_{[a_1 | \pm [I_1 I_2 |} G_{| a_2] | \pm [I_3 I_4]} + \frac{1}{4!} \epsilon_{I_1 \dots I_4}^{J_1 \dots J_4} \Lambda_{[a_1 | \pm J_1 J_2} G_{| a_2] | \pm J_3 J_4} \\
\delta G_{a_1 a_2 S^J K} &= \sum_{\pm} \mp 8i \Lambda_{[a_1 | \pm L (K |} G_{| a_2] \pm L}^{J)} \pm i \delta_K^J \Lambda_{[a_1 | \pm L M} G_{| a_2] \pm L}^M, \\
\delta G_{a_1 a_2 A^J K} &= \sum_{\pm} \pm 8i \Lambda_{[a_1 | \pm L (K |} G_{| a_2] \mp L}^{J)} \\
\delta \hat{G}_{a_1 a_2} &= \sum_{\pm} \mp 4i \Lambda_{(a_1 | \pm J_1 J_2} G_{| a_2) \mp J_1 J_2} \tag{3.25}
\end{aligned}$$

Let us now turn our attention to the transformation of the part of the Cartan form in the direction of the l_1 representation, that is \mathcal{V}_l . At lowest level, the transformation of equation (3.13) implies, using equation (3.17) that

$$E_{\Pi}^{A'} = E_{\Pi}^B D(h)_B^A, \quad \text{and for the inverse} \quad (E^{-1})_A^{\Pi'} = D(h^{-1})_A^B (E^{-1})_B^{\Pi} \tag{3.26}$$

if we define $h^{-1} L_A h = D(h)_A^B L_B$. At lowest levels this implies the local transforms

$$\begin{aligned}
\delta E_{\Pi}^a &= -4 E_{\Pi}^{-I_1 I_2} \Lambda_{a+}^{I_1 I_2} - 4 E_{\Pi}^{+I_1 I_2} \Lambda_{a-I_1 I_2}, \quad \delta E_{\Pi}^{\pm I_1 I_2} = \Lambda_{a \pm I_1 I_2} E_{\Pi}^b, \dots \\
\delta (E^{-1})_a^{\Pi} &= -\Lambda_{a+I_1 I_2} (E_+^{-1})_{I_1 I_2}^{\Pi} - \Lambda_{a-I_1 I_2} (E_-^{-1})_{I_1 I_2}^{\Pi} \\
\delta (E_{\pm}^{-1})^{I_1 I_2 \Pi} &= 4 \Lambda_{a \mp I_1 I_2} (E^{-1})_b^{\Pi}, \dots \tag{3.27}
\end{aligned}$$

In the above we have written the Cartan forms as forms and were we to write them out explicitly we would write \mathcal{V}_E as $dz^{\Pi} G_{\Pi, \bullet} R^{\bullet}$, where \bullet denotes a generic E_{11} index, and \mathcal{V}_l as $dz^{\Pi} E_{\Pi}^A l_A$. Put another way we have suppressed their world index Π . Even though the Cartan forms are invariant under the rigid transformations, E_{Π}^A and $G_{\Pi, \bullet}$ are not as the transformation of z^{Π} of equation (3.8) implies a corresponding inverse transformation acting on the Π index of these two objects. Thus E_{Π}^A transforms under a local transformation on its A index and by the inverse of the coordinate transformation on its Π index. As such we can think of it as a generalised vielbein. We can rewrite the Cartan form of $E_{11} \otimes_s l_1$ as

$$\mathcal{V} = g^{-1} dg = dz^{\Pi} E_{\Pi}^A (L_A + G_{A, *} R^*) \tag{3.28}$$

where $G_{A, \bullet} = (E^{-1})_A^{\Pi} G_{\Pi, \bullet}$. At low levels $(E^{-1})_A^{\Pi}$ is the inverse of the matrix of equation (3.19). Clearly $G_{A, \bullet}$ is inert under rigid transformations, but it transforms under local

transformations as in equation (3.25) on its \bullet index and as the inverse generalised vielbein on its A index, that is as in equation (3.27). The latter transformation can be written as

$$\delta G_{a,\bullet} = -\Lambda_{a+I_1 I_2} G_{+I_1 I_2,\bullet} - \Lambda_{a-I_1 I_2} G_{-I_1 I_2,\bullet}, \quad \delta G_{\pm I_1 I_2,\bullet} = 4\Lambda_{b\mp I_1 I_2} G_{b,\bullet}, \dots \quad (3.29)$$

Of course, the full local transformation is the sum of that given in equations (3.27) and (3.29).

The $SU(8)$ variations of the Cartan forms is given on their E_{11} index by taking $h = 1 - \Lambda^{I_1 \dots I_4} R^{I_1 \dots I_4}$ in equation (3.10) and using equations (2.42), (2.43), (2.46) and (2.48); the result is

$$\begin{aligned} \delta G_{\diamond, I}^J &= 6\Lambda_{K_1 \dots K_3 J} G_{\diamond, K_1 \dots K_3 I} + 6\Lambda_{K_1 \dots K_3 I} G_{\diamond, K_1 \dots K_3 J}, \quad \delta G_{\diamond, J_1 \dots J_4} = -16G_{\diamond, [J_1 |}^K \Lambda_{K | J_1 \dots J_4]} \\ \delta G_{\diamond, \pm a I_1 I_2} &= \pm 2i\Lambda_{J_1 J_2 I_1 I_2} G_{\diamond, \pm a J_1 J_2} \\ \delta G_{\diamond, a_1 a_2 S}^I J &= -\frac{8}{3}\Lambda_{[I | K_1 K_2 K_3} G_{\diamond, a_1 a_2 + K_1 K_2 K_3 | J]} \\ \delta G_{\diamond, a_1 a_2 A}^I J &= -\frac{8}{3}\Lambda_{[I | K_1 K_2 K_3} G_{\diamond, a_1 a_2 - K_1 K_2 K_3 | J]} \\ \delta G_{\diamond, a_1 a_2 +}^{I_1 \dots I_4} &= -4\Lambda_{[I_1 I_2 I_3 | K} G_{\diamond, a_1 a_2 S}^K | I_4] \\ \delta G_{\diamond, a_1 a_2 -}^{I_1 \dots I_4} &= -4\Lambda_{[I_1 I_2 I_3 | K} G_{\diamond, a_1 a_2 A}^K | I_4] \end{aligned} \quad (3.30)$$

While the Cartan forms transform under $SU(8)$ on their l_1 , or A index as follows

$$\delta G_{\pm I_1 I_2, \bullet} = \mp 2i\Lambda_{I_1 I_2 J_1 J_2} G_{\pm J_1 J_2, \bullet} \quad (3.31)$$

4 The equations of motion

In this section we will construct the invariant equations of motion using the variations found in the last section. We found at the end of section three that the Cartan forms referred to the tangent space, see equation (3.28), are inert under the rigid E_{11} transformations and only transform under local $I_c(E_{11})$ transformations. Let us denote the Cartan forms in \mathcal{V}_E , when referred to tangent space, by $G_{\diamond, \bullet}$ where \bullet is a generic E_{11} index and \diamond is a generic form index referred to the tangent space using the generalised vielbein E_A^Π , in other words \diamond is the index used to label the l_1 representation. The dynamics is by definition just a set of equations which are invariant under the local and rigid transformations of the non-linear realisation. Thus if we construct the dynamics out of $G_{\diamond, \bullet}$ we need only worry about the local transformations. Hence to find the dynamics is just a problem in group theory. However our knowledge of the properties of $I_c(E_{11})$ is limited and so, for the time being, we must carry this calculation out level by level. We will demand that the equations of motion are first order in derivatives and so first order in $G_{\diamond, \bullet}$. This is a special feature of E_{11} reflecting the fact that E_{11} is a duality symmetry generalising electromagnetic duality.

The level zero Cartan involution invariant subalgebra is $SO(4) \times SU(8)$ and so we can choose to classify the equations of motion by representations of this algebra. The Cartan involution invariant subalgebra is generated by the level zero sector and the generators

$S_{a\pm I_1 I_2}$ and so to check the invariance under the full non-linear realisation we need only check that the equations of motion are inert under these transformations. As such we will write down all terms, with arbitrary coefficients, in the chosen representation of $SO(4) \times SU(8)$, up to the level being studied, and then vary them under the transformations of the Cartan involution invariant subalgebra; that is, the the $SU(8)$ transformations given in equations (3.30) and (3.31) and the $S_{a\pm I_1 I_2}$ transformations given in equations (3.25) and (3.29).

Let us begin with the equation of motion whose terms belongs to the 6-dimensional representation of $SO(4)$, that is two antisymmetrised indices, and the 28 ($\bar{28}$)-dimensional representation of $SU(8)$. While the representations of $SO(4)$ are obvious the same is not always true for those of $SU(8)$, at least in the formalism we are using. However, in constructing objects that transform as representations of $SU(8)$ we can be guided by the well known action of the $SO(8)$ subgroup. The most obvious such terms are those Cartan forms whose \bullet index, that is E_{11} index, carries, at least in part, the 28 ($\bar{28}$)-dimensional representation of $SU(8)$ and whose \diamond index is just the four dimensional representation of $GL(4)$, that is, the object $G_{[a_1, a_2] + I_1 I_2} (G_{a_1, a_2 - I_1 I_2})$. However, we can also consider the Cartan forms whose \bullet index contains the 70-dimensional representation of $SU(8)$, that is use the objects $G_{\diamond, a_1 a_2 + I_1 \dots I_4}$ and $G_{\diamond, a_1 a_2} S^I J$, and whose \diamond index, that is l_1 index, belongs to the 28 ($\bar{28}$) representation of $SU(8)$, that is $G_{I_1 I_2 \pm, \bullet}$. We can then form the 28 ($\bar{28}$) of $SU(8)$ using the tensor product rules $28 \times 70 = \bar{28} + \dots$ ($\bar{28} \times 70 = 28 + \dots$). Thus we consider the sum of the two terms

$$G_{\pm J_1 J_2, a_1 a_2 +}^{J_1 J_2 I_1 I_2}, \quad G_{\pm [I_1 | K, a_1 a_2} S^K_{| I_2]} \quad (4.1)$$

Using the $SU(8)$ variations of the Cartan forms of equations (3.30) and (3.31) we find that the combination

$$\Delta G_{\pm a_1 a_2 70 I_1 I_2} \equiv G_{\pm J_1 J_2, a_1 a_2 +}^{J_1 J_2 I_1 I_2} \pm i G_{\pm [I_1 | K, a_1 a_2} S^K_{| I_2]} \quad (4.2)$$

transforms under $SU(8)$ as

$$\delta(\Delta G_{\pm a_1 a_2 70 I_1 I_2}) = \pm 2i \Lambda_{I_1 I_2 K_1 K_2} \Delta G_{\pm a_1 a_2 70 K_1 K_2} \quad (4.3)$$

that is like the 28 ($\bar{28}$)-dimensional representation of $SU(8)$ and so like $G_{\diamond, a \pm I_1 I_2}$. The use of the subscript 70 reminds the reader of where this term originated and it will be used to distinguish this term from a similar term that we will also now construct.

We can also form the 28 ($\bar{28}$) representation by taking the \bullet index to be the 63-dimensional representation of $SU(8)$ instead of the 70-dimensional representation and using the tensor product rules $28 \times 63 = 28 + \dots$ ($\bar{28} \times 63 = \bar{28} + \dots$). As a result we consider the terms

$$G_{\pm J_1 J_2, a_1 a_2 -}^{J_1 J_2 I_1 I_2}, \quad G_{\pm [I_1 | K, a_1 a_2} A^K_{| I_2]} \quad (4.4)$$

Proceeding as before we find that the combination

$$\Delta G_{\pm a_1 a_2 63 I_1 I_2} \equiv G_{\mp J_1 J_2, a_1 a_2 -}^{J_1 J_2 I_1 I_2} \mp i G_{\mp [I_1 | K, a_1 a_2} A^K_{| I_2]} \quad (4.5)$$

transforms as

$$\delta(\Delta G_{\pm a_1 a_2 63 I_1 I_2}) = \pm 2i \Lambda_{I_1 I_2 K_1 K_2} \Delta G_{\pm a_1 a_2 63 K_1 K_2} \quad (4.6)$$

that is as the 28 ($\bar{28}$)-dimensional representation of SU(8).

Finally, we can write down all possible terms in the equation of motion that transforms as the 6-dimensional representation of SO(4) and the 28 ($\bar{28}$)-dimensional representation of SU(8); taking arbitrary coefficients they are given by

$$\begin{aligned} G_{[a_1, a_2] \pm I_1 I_2} + i \frac{e_{1\pm}}{2} \epsilon_{a_1 a_2}{}^{b_1 b_2} G_{b_1, b_2 \pm I_1 I_2} + e_{70\pm} \Delta G_{\pm a_1 a_2 70 I_1 I_2} + e_{63\pm} \Delta G_{\pm a_1 a_2 63 I_1 I_2} \\ + i \frac{e_{1\pm}}{2} \epsilon_{a_1 a_2}{}^{b_1 b_2} (e'_{70\pm} \Delta G_{\pm a_1 a_2 70 I_1 I_2} + e'_{63\pm} \Delta G_{\pm a_1 a_2 63 I_1 I_2}) + \dots = 0 \end{aligned} \quad (4.7)$$

where $+\dots$ mean terms at level greater than two in the fields and derivatives with respect to the coordinates that are greater than level zero.

Varying equation (4.7) under a local transformation, but keeping only terms that contain the Cartan forms $G_{a_1, a_2 \pm I_1 I_2}$, we find that the first term leads to a term of the form

$$4G_{[a_1], [a_2] + I_1 I_2 J_1 J_2} \Lambda_{b \mp J_1 J_2} \quad (4.8)$$

as well as other terms. We can rewrite this term as

$$2G^b_{, a_1 a_2 + I_1 I_2 J_1 J_2} \Lambda_{b \mp J_1 J_2} - 6G_{[b, a_1 a_2] + I_1 I_2 J_1 J_2} \Lambda^b_{\mp J_1 J_2} \quad (4.9)$$

The first term can be canceled if we choose the constant in equation (4.7) to be given by $e_{70\pm} = -\frac{1}{2}$. The second term is of the form of a field strength and, as we will see, it is required to find the original equation again after the variation. We note that although the local variations of the Cartan forms do not lead to field strengths for the gauge fields there are allowed terms in the equation of motion that involve derivatives with respect to the extra coordinates which cancel the non-field strength terms. Proceeding in the same way for similar variations one finds that $e_{63\pm} = -\frac{1}{2}$. Thus we find that, for these coefficients, the variation of equation (4.7) is invariant if we discard variations that involve other fields. Collecting these results we find that equation (4.7) can be rewritten as

$$\mathcal{G}_{[a_1, a_2] \pm I_1 I_2} \pm \frac{i}{2} \epsilon_{a_1 a_2}{}^{b_1 b_2} \mathcal{G}_{b_1, b_2 \pm I_1 I_2} = 0 \quad (4.10)$$

where

$$\mathcal{G}_{[a_1, a_2] \pm I_1 I_2} \equiv G_{[a_1, a_2] \pm I_1 I_2} - \frac{1}{2} \Delta G_{\pm a_1 a_2 70 I_1 I_2} - \frac{1}{2} \Delta G_{\pm a_1 a_2 63 I_1 I_2} \quad (4.11)$$

We recognise these as the correct equations of motion of the gauge fields once one acts with another derivative and takes the fields to depend only on the usual coordinates of four-dimensional space-time.

When carrying out the variations in this section we consider only terms in the variations that have derivatives with respect to the usual coordinates of space-time. As a result one finds new equations that contain only terms with space-time derivatives. However, when we vary these equations the result is sensitive, by using equation (3.29), to terms in

the original equation that contain derivatives with respect to the generalised coordinates, that is, the derivative $E_{\pm I_1 I_2}{}^\Pi \partial_\Pi$.

We now take all the other variations of equation (4.7) under equations (3.25) and (3.29), except those involving the fields of gravity and dual gravity. We find that it leads to the equations

$$G_{a, I_1 I_2 J_1 J_2} - \epsilon_{ab_1 b_2 b_3} G_{b_1, b_2 b_3 +}{}^{I_1 I_2 J_1 J_2} = 0 \quad (4.12)$$

$$G_{a, I J} - \epsilon_{ab_1 b_2 b_3} G_{b_1, b_2 b_3 S}{}^I{}_J = 0 \quad (4.13)$$

In carrying out this calculation one must set to zero the coefficients of the parameters $\Lambda_{a+I_1 I_2}$ and $\Lambda_{a-I_1 I_2}$ as well as the independent $\text{SO}(1,3)$ tensor structures and in doing so one finds two copies of the above equations that are only consistent if $e_{1\pm}^2 = 1$; in the above equation we have chosen $e_{1\pm} = \pm 1$.

Varying the equations of motion for the scalars (4.12) and (4.13) under the local $I_c(E_{11})$ transformations one finds the vector equation of motion of equation (4.10). However, as explained above, it is in carrying out this step one finds the contributions in equations (4.12) and (4.13) that contain the derivatives with respect to the Lorentz scalar coordinates and the actual equations now read

$$\mathcal{G}_{a, I_1 I_2 J_1 J_2} - \epsilon_{ab_1 b_2 b_3} \mathcal{G}_{b_1, b_2 b_3 +}{}^{I_1 I_2 J_1 J_2} = 0 \quad (4.14)$$

$$\mathcal{G}_{a, I J} - \epsilon_{ab_1 b_2 b_3} \mathcal{G}_{b_1, b_2 b_3 S}{}^I{}_J = 0 \quad (4.15)$$

where

$$\mathcal{G}_{a, I_1 I_2 J_1 J_2} = G_{a, I_1 I_2 J_1 J_2} + 6\Delta G_{a, I_1 I_2 J_1 J_2}, \quad \mathcal{G}_{a, I J} = G_{a, I J} + 6\Delta G_{a, I J} \quad (4.16)$$

and

$$\begin{aligned} \Delta G_{a, I_1 I_2 J_1 J_2} = & +\frac{i}{2}(G_{-[I_1 I_2], a+|I_3 I_4]} + \frac{1}{4!}\epsilon_{I_1 \dots I_4 K_1 \dots K_4} G_{-[K_1 K_2], a+|K_3 K_4]}) \\ & -\frac{i}{2}(G_{+[I_1 I_2], a-|I_3 I_4]} + \frac{1}{4!}\epsilon_{I_1 \dots I_4 K_1 \dots K_4} G_{+[K_1 K_2], a-|K_3 K_4]}) , \\ \Delta G_{a, I J} = & +\frac{1}{24}(G_{-IK, a+KJ} + G_{-JK, a+KI} - \frac{1}{4}\delta_J^I G_{-LK, a+KL} \\ & + G_{+IK, a-KJ} + G_{+JK, a-KI} - \frac{1}{4}\delta_J^I G_{+LK, a-KL}) \end{aligned} \quad (4.17)$$

One can verify that the combinations $\Delta G_{a, I_1 I_2 J_1 J_2}$ and $\Delta G_{a, I J}$ transform as the 70-dimensional representations of $\text{SU}(8)$, that is, as $G_{a, I_1 I_2 I_3 I_4}$ and $G_{a, I J}$ do.

Taking another derivative acting on equations (4.14) and (4.15) we find the equations of motion for the scalars of four dimensional maximal supergravity provided we again take the fields to depend only on just the usual coordinates of four dimensional space-time.

In varying equation (4.10) we also find the equations

$$G_{b_1, b_2 b_3 -}{}^{I_1 I_2 I_3 I_4} = 0 \quad (4.18)$$

$$G_{b_1, b_2 b_3 A}{}^I{}_J = 0 \quad (4.19)$$

These equations are expected as the fields in the non-linear realisation which are dual to the scalars belong to the 133 of E_7 , however, there are only 70 scalars as they belong to the non-linear realisation of E_7 with local subgroup $SU(8)$, equivalently the coset $E_7/SU(8)$. The variations of these equations will be discussed later.

Thus one finds an infinite set of equations of motion that are invariant under the symmetries of the non-linear realisation; the lowest two equations being the equations of motion for the gauge fields, equations (4.10), and scalars equations (4.14) and (4.15). These latter equations are equivalent to the equations for the four dimensional maximal supergravity theory provided we take the fields not to depend on the Lorentz scalar coordinates. We note that these equations are uniquely determined by the symmetries once we pick the Lorentz and $SU(8)$ character of one of them.

We now consider equations whose Lorentz and $SU(8)$ character do not occur in the above infinite set. In particular we consider the equation that is a Lorentz scalar but transforms under the 28 ($\bar{28}$) of $SU(8)$. Up to the level at which we are working the only possible terms that this can contain are

$$G_{a,}{}^a{}_{\pm I_1 I_2}, \quad G_{\pm J_1 J_2,}{}^{J_1 J_2 I_1 I_2}, \quad G_{\pm [I_1 | K,}{}^K{}_{| I_2]} \quad (4.20)$$

Varying under the local transformations of section three we find that this equation will be invariant if it takes the form

$$G_{a,}{}^a{}_{\pm I_1 I_2} \pm \frac{i}{2} G_{\pm J_1 J_2,}{}^{J_1 J_2 I_1 I_2} + \frac{1}{2} G_{\pm [I_1 | K,}{}^K{}_{| I_2]} = 0 \quad (4.21)$$

and we also impose the additional equations

$$G_{a,}{}^b{}_{a+I_1 \dots I_4} + \frac{3}{2} G_{-[I_1 I_2 |, b+| I_3 I_4]} + \frac{3}{2} G_{+[I_1 I_2 |, b-| I_3 I_4]} = 0 \quad (4.22)$$

$$G_{a,}{}^b{}_{a-I_1 \dots I_4} + \frac{3}{2} G_{-[I_1 I_2 |, b-| I_3 I_4]} + \frac{3}{2} G_{+[I_1 I_2 |, b+| I_3 I_4]} = 0 \quad (4.23)$$

$$G_{a,}{}^b{}_{aS}{}^J{}_K - iG_{-L(K|b, +L|J)} + iG_{+L(K|b, -L|J)} = 0 \quad (4.24)$$

and

$$G_{a,}{}^b{}_{aA}{}^J{}_K + iG_{-L(K|b, -L|J)} - iG_{+L(K|b, +L|J)} = 0 \quad (4.25)$$

One can verify that the local variation of equations (4.22) to (4.25) leads to equation (4.20). Thus one finds another infinite tower of invariant equations which are uniquely specified by the symmetries of the non-linear realisation.

The equations (4.21-4.25) can be thought of as gauge conditions, if one sets the dependence of the fields to be just that of the coordinates of the usual four dimensional space-time. Of course one does not have to actually adopt these latter equations and one can just take the equations (4.10), (4.14), (4.15) and their higher level analogues.

We conclude this section with some incomplete results on the higher level equations of motion. By varying the field equation for the gauge fields we found equations (4.18) and (4.19). Varying the first equation under the $I_c(E_{11})$ local variations we find that

$$8 \sum_{\pm} (-\Lambda_{b \pm L[J] G_{b, a \mp L[I]} + \Lambda_{b \pm L[J] G_{a, b \mp L[I]}) = 0 \quad (4.26)$$

The first term can be canceled by adding to the left-hand side of equation (4.18) the term

$$2 \sum_{\pm} G_{\mp L[J, a \mp L|I]} \quad (4.27)$$

However, the second term can be canceled by adding the term

$$Q_{a,}{}^I{}_J \quad (4.28)$$

Where $Q_{\diamond, \bullet}$ are the Cartan forms belonging to $I_c(E_{11})$. This has a local transformation which given by

$$\delta Q_{a,}{}^I{}_J = -4 \sum_{\pm} \Lambda_{a \pm L[J|} Q_{a, b \mp L|I]} \quad (4.29)$$

As we noted in the gauge in which we are working $Q_{a, b \mp L|I]} = G_{a, b \mp L|I]}$. However, $Q_{\diamond, \bullet}$ does not transform homogeneously as it has a $h^{-1}dh$ part. As such once we add terms of this type the equations of motion only hold modulo this inhomogeneous term. Covariant equations can be found by acting with a derivative in an appropriate way. The resulting equation which replaces equation (4.18) is

$$\epsilon^{ab_1 b_2 b_3} G_{b_1, b_2 b_3 A}{}^I{}_J + 2 \sum_{\pm} G_{\mp L[J|, a \mp L|I]} + Q_{a,}{}^I{}_J = 0 \quad (4.30)$$

A similar analysis applies to equation (4.19) which is now replaced by the equation

$$\epsilon^{ab_1 b_2 b_3} G_{b_1, b_2 b_3 -}{}^{I_1 \dots I_4} - 3i \sum_{\pm} \mp G_{\mp [I_1 I_2|, a \pm |I_3 I_4]} + \frac{1}{2} Q_{a, I_1 \dots I_4} = 0 \quad (4.31)$$

The above steps are required in any non-linear realisation that is constructed from a Kac-Moody algebra and involves scalars and has dual fields. To illustrate this let us consider that the theory contains two scalars that belong to the non-linear realisation of $SL(2, R)$ which is part of the larger Kac-Moody algebra. The theory will also contain dual fields which carry $D-2$ space-time indices, if D is the dimension of space-time, and belong to the adjoint representation of $SL(2, R)$. These lead to three field strengths which transform in the adjoint representation of $SL(2, R)$. However, only two of these are related to the two scalars by a duality relation. This is possible as the Cartan forms transform under the local symmetry which for $SL(2, R)$ is $SO(2)$. While two of the Cartan forms are doublets the remainder is a singlet and this can be set to zero, at least as far as the subalgebra $SL(2, R)$ is concerned. The extra $D-2$ form field arises as the $D-2$ forms must belong to a multiplet of $SL(2, R)$ while the scalars belong to the coset $SL(2, R)/SO(2)$. However, the variation of this field under the other transformations of the local subalgebra involves the other fields from the non-linear realisation and these must be cancelled, hopefully in the way explained above. One of the simplest contexts in which to think about this problem is the IIB theory which has an obvious $SL(2, R)$ subalgebra.

Varying the gauge field equation (4.10) but now keeping the gravity and dual gravity fields we find that its real and imaginary parts are the same and are given by

$$G_{[a_1|,}{}^b{}_{|a_2]} + \frac{1}{4} \epsilon_{a_1 a_2}{}^{b_1 b_2} \hat{G}_{b_1,}{}^b{}_{b_2} = 0 \quad (4.32)$$

However to find the full equation one must vary this under the local symmetry and then add the terms that contain derivatives with respect to the generalised coordinates and the "Q" terms. The latter are the $Q_a{}^b{}_c$, that is the parts of the Cartan forms associated with the Lorentz algebra. As a result the gravity equation will only hold up to local Lorentz transformations which include a term which contains the derivative of the Lorentz parameter. We note that the formulation of the correct gravity equation has some features that are similar to those for the scalar and this should increase the prospect that the solution can be found within the context of the Kac-Moody algebra.

The resulting equation for gravity and those of equations (4.30) and (4.31) are still being studied and the author expects to write a more complete account in a subsequent publication. We also hope to report on the significance of the second set of equations.

5. Discussion

One can view the above computation from a slightly different perspective. We have considered objects that are first order in the derivatives of the generalised space-time. What we have shown, to the level to which we are working, is that the right-hand sides of equations (4.10), (4.14) and (4.15) vary into each other under the local symmetry $I_c(E_{11})$ and so transform covariantly. Similarly, the left-hand side of equations (4.21-4.25) vary into each other under the local symmetry. Since the rigid symmetry is automatically encoded in the way we have performed the computation it follows that we have found two sets of expressions each of which transform covariantly under all the symmetries of the non-linear realisation. Furthermore the two sets are uniquely determined by the non-linear realisation, that is, the properties of the E_{11} Kac-Moody algebra and its first fundamental representation l_1 . The only assumption we have made is that the objects we consider are first order in the generalised space-time derivatives.

One is not forced to set either of these two sets of expressions to zero, however, if one sets the first set to zero then one will find an infinite number of equations the first two of which correctly describe the equations of motion of the scalars and the gauge fields once we consider the fields to depend only on the usual coordinates of four dimensional space-time.

We can state the result in a more group theoretic manner. The Cartan forms in the coset direction, that is the P , carry a representation of $I_c(E_{11})$; the transformations acting on the indices that are inherited from the adjoint representation of E_{11} as well as the indices that arise from the l_1 representation. However, this representation is not irreducible as there exists, at least at low levels, an involution operator on the representation which can be used to define the irreducible components. This involution includes the action of the epsilon symbol of the usual space-time and it maps fields to their duals. Thus it is a generalisation of our usual notion of electromagnetic duality. It would be good to understand the representation carried by the Cartan forms and the involution in a more abstract way as this could lead to a more efficient way of computing the equations of motion rather than the order by order method used in this paper.

We note that when we discard coordinates from the generalised space-time except those of the usual four-dimensional space-time then the equations are gauge invariant and are unique so we did not have to adjust any constants in order to achieve this. This is in contrast to previous such computations in the early papers on E_{11} , and for example

in references [24,25], where one found the equations of motion, or Lagrangian, were only determined up to some constants. This problem was addressed in the earliest E_{11} papers by demanding that the equations also be conformally invariant, an idea which was first used in reference [19], or by simply demanding gauge symmetry as in [24,25]. The difference with the calculation of this paper is that one has implemented the symmetries of the non-linear realisation at a higher level and in particular the local symmetries which were often taken to be just those at the very lowest level, that is, just the Lorentz group.

The results found in this paper are similar to the calculation of the $E_{11} \otimes_s l_1$ non-linear realisation in ten and eleven dimensions given in references [29,30] and [31] respectively. Also in these papers one considered quantities that were first order in the derivatives with respect to the generalised space-time and one found only two unique sets which transform covariantly into themselves under the symmetries of the non-linear realisation. Setting one of these sets to zero leads to the equations of motion of the corresponding supergravity. The way the results in these papers were phrased were a bit different, but it is equivalent to the statement just made.

In the full non-linear realisation the field equations will depend on the higher level fields which arise from the E_{11} part of the non-linear realisation and they will lead to effects which it would be interesting to study. In particular we already know that the three form fields at level four will lead to the gauged supergravities in four dimensions [12,36].

It is striking, at least to this author, that the $E_{11} \otimes l_1$ non-linear realisation leads essentially uniquely to the correct equations for the scalars and gauge fields and an equation, yet to be fully formulated, for the gravity that has many of the correct features. Indeed if the latter equation were to turn out to be correct then the E_{11} conjecture would be proven. The present paper provides a good arena to see if this is indeed the case.

The equations contain derivatives with respect to the higher level coordinates belonging to the generalised space-time. Just setting to zero the derivatives with respect to all the coordinates except those of the usual four dimensional space-time is not a satisfactory step. It is difficult to believe that the additional coordinates, beyond those of the usual four dimensional space-time, are not there for a reason. Indeed, as we have already noted the higher level coordinates do play an important role in the formulation of the gauged supergravities [13]. However, it remains to implement a truly satisfactory, physically motivated, procedure that carries out the required radical reduction in the number of coordinates. We note that for the $E_{11} \otimes l_1$ non-linear realisation, at lowest level, in the decomposition that leads to the IIA theory the reduction has been found to occur by considering the first quantised theory [38]. While the full twenty dimensional generalised space-time occurs in the first quantised theory its quantisation to find the field theory requires that half of the coordinates are eliminated. Thus the quantisation breaks the manifest $SO(10,10)$ symmetry, however, if one takes into account all possible ways of choosing the ten dimensional slice of space-time that remains then the theory should possess the full symmetry, albeit in a hidden way. It would be interesting to extend these results to the full $E_{11} \otimes l_1$ non-linear realisation.

Acknowledgment

I wish to thank Paul Cook and Michael Fleming for discussions and the SFTC for support from grant number ST/J002798/1.

Appendix A The $E_{11} \otimes_s l_1$ algebra

This appendix is designed to equip the reader with the E_{11} material required to understand this paper. Rather than explain the theory behind Kac-Moody algebras we will present the required results. We first give the E_{11} algebra in the decomposition appropriate to eleven dimensions, that is, we decompose the E_{11} algebra into representations of A_{10} , or $SL(11)$, representations [10,16]. This algebra is found by deleting node eleven in the Dynkin Diagram of E_{11} given below

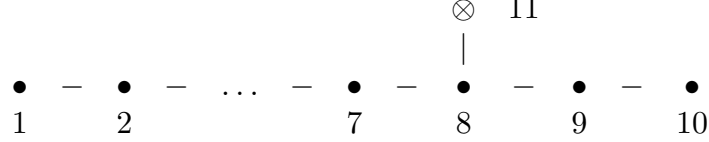


Fig 1. The E_{11} Dynkin diagram

The way one constructs this algebra from the definition of E_{11} as a Kac-Moody algebra in terms of representations of $SL(11)$ is discussed, for example, in [17]. For the calculation in this paper one does not need to understand all the subtleties of this construction and the parts of the algebra that are needed are given below. The generators can be classified according to a level which is associated with the decomposition associated with the deletion of node eleven. At level zero we have the algebra $GL(11)$ with the generators K^a_b , $a, b = 1, \dots, 11$ and at level one and minus one the rank three generators R^{abc} and R_{abc} respectively. The generators at level two and minus two are $R^{a_1 \dots a_6}$ and $R_{a_1 \dots a_6}$ respectively, while those at levels three and minus three are $R^{a_1 \dots a_8, b}$ and $R_{a_1 \dots a_8, b}$ respectively. The level is just the number of upper minus lower indices divided by three. For a discussion giving the more abstract definition of level which relates it to the deletion of node eleven see for example references [17] or [38].

The E_{11} algebra at levels zero and up three is given by [10,16]

$$[K^a_b, K^c_d] = \delta_b^c K^a_d - \delta_d^a K^c_b, \quad (A.1)$$

$$[K^a_b, R^{c_1 \dots c_6}] = \delta_b^{c_1} R^{ac_2 \dots c_6} + \dots, \quad [K^a_b, R^{c_1 \dots c_3}] = \delta_b^{c_1} R^{ac_2 c_3} + \dots, \quad (A.2)$$

$$[K^a_b, R^{c_1 \dots c_8, d}] = (\delta_b^{c_1} R^{ac_2 \dots c_8, d} + \dots) + \delta_b^d R^{c_1 \dots c_8, a}. \quad (A.3)$$

and

$$[R^{c_1 \dots c_3}, R^{c_4 \dots c_6}] = 2R^{c_1 \dots c_6}, \quad [R^{a_1 \dots a_6}, R^{b_1 \dots b_3}] = 3R^{a_1 \dots a_6} [b_1 b_2, b_3], \quad (A.4)$$

where $+\dots$ means the appropriate anti-symmetrisation.

The E_{11} level zero and negative level generators up to level minus three obey the relations

$$[K^a_b, R_{c_1 \dots c_3}] = -\delta_{c_1}^a R_{bc_2 c_3} - \dots, \quad [K^a_b, R_{c_1 \dots c_6}] = -\delta_{c_1}^a R_{bc_2 \dots c_6} - \dots, \quad (A.5)$$

$$[K^a_b, R_{c_1 \dots c_8, d}] = -(\delta_{c_1}^a R_{bc_2 \dots c_8, d} + \dots) - \delta_d^a R_{c_1 \dots c_8, b}. \quad (A.6)$$

$$[R_{c_1 \dots c_3}, R_{c_4 \dots c_6}] = 2R_{c_1 \dots c_6}, \quad [R_{a_1 \dots a_6}, R_{b_1 \dots b_3}] = 3R_{a_1 \dots a_6} [b_1 b_2, b_3], \quad (A.7)$$

Finally, the commutation relations between the positive and negative generators are given by

$$\begin{aligned}
[R^{a_1 \dots a_3}, R_{b_1 \dots b_3}] &= 18\delta_{[b_1 b_2}^{[a_1 a_2} K^{a_3]}_{b_3]} - 2\delta_{b_1 b_2 b_3}^{a_1 a_2 a_3} D, \quad [R_{b_1 \dots b_3}, R^{a_1 \dots a_6}] = \frac{5!}{2}\delta_{b_1 b_2 b_3}^{a_1 a_2 a_3} R^{a_4 a_5 a_6} \\
[R^{a_1 \dots a_6}, R_{b_1 \dots b_6}] &= -5! \cdot 3 \cdot 3 \delta_{[b_1 \dots b_5}^{[a_1 \dots a_5} K^{a_6]}_{b_6]} + 5! \delta_{b_1 \dots b_6}^{a_1 \dots a_6} D, \\
[R_{a_1 \dots a_3}, R^{b_1 \dots b_8, c}] &= 8 \cdot 7 \cdot 2 (\delta_{[a_1 a_2 a_3}^{[b_1 b_2 b_3} R^{b_4 \dots b_8]c} - \delta_{[a_1 a_2 a_3}^{[b_1 b_2 | c|} R^{b_3 \dots b_8]}) \\
[R_{a_1 \dots a_6}, R^{b_1 \dots b_8, c}] &= \frac{7! \cdot 2}{3} (\delta_{[a_1 \dots a_6}^{[b_1 \dots b_6} R^{b_7 b_8]c} - \delta_{[a_1 \dots a_6}^{c [b_1 \dots b_5} R^{b_6 b_7 b_8]}) \tag{A.8}
\end{aligned}$$

where $D = \sum_b K^b_b$, $\delta_{b_1 b_2}^{a_1 a_2} = \frac{1}{2}(\delta_{b_1}^{a_1} \delta_{b_2}^{a_2} - \delta_{b_1}^{a_2} \delta_{b_2}^{a_1}) = \delta_{b_1}^{[a_1} \delta_{b_2}^{a_2]}$ with similar formulae when more indices are involved.

We also need the fundamental representation of E_{11} associated with node one, denoted by l_1 . By definition this is the representation with highest weight Λ_1 which obeys $(\Lambda_1, \alpha_a) = \delta_{a,1}$, $a = 1, 2, \dots, 11$ where α_a are the simple roots of E_{11} . In the decomposition to $Sl(11)$, corresponding to the deletion of node eleven, one finds that the l_1 representation contains the objects P_a , Z^{ab} and $Z^{a_1 \dots a_5}$, $a, b, a_1, \dots = 1, \dots, 11$ corresponding to levels zero, one and two respectively. We have taken the first object, i.e. P_a , to have level zero by choice. Taking these to be generators belong to a semi-direct product algebra with those of E_{11} , denoted by $E_{11} \otimes_s l_1$, their commutation relations with the level one generators of E_{11} are given by [16]

$$\begin{aligned}
[R^{a_1 a_2 a_3}, P_b] &= 3\delta_b^{[a_1} Z^{a_2 a_3]}, \quad [R^{a_1 a_2 a_3}, Z^{b_1 b_2}] = Z^{a_1 a_2 a_3 b_1 b_2}, \\
[R^{a_1 a_2 a_3}, Z^{b_1 \dots b_5}] &= Z^{b_1 \dots b_5 [a_1 a_2, a_3]} + Z^{b_1 \dots b_5 a_1 a_2 a_3} \tag{A.9}
\end{aligned}$$

These equations define the normalisation of the generators of the l_1 representation. The commutators of the generators of the l_1 representation with those of $GL(11)$ are given by

$$\begin{aligned}
[K^a_b, P_c] &= -\delta_c^a P_b + \frac{1}{2}\delta_b^a P_c, \quad [K^a_b, Z^{c_1 c_2}] = 2\delta_b^{[c_1} Z^{a|c_2]} + \frac{1}{2}\delta_b^a Z^{c_1 c_2}, \\
[K^a_b, Z^{c_1 \dots c_5}] &= 5\delta_b^{[c_1} Z^{a|c_2 \dots c_5]} + \frac{1}{2}\delta_b^a Z^{c_1 \dots c_5} \tag{A.10}
\end{aligned}$$

The commutation relations with the level two generators of E_{11} are given by

$$[R^{a_1 \dots a_6}, P_b] = -3\delta_b^{[a_1} Z^{\dots a_6]}, \quad [R^{a_1 \dots a_6}, Z^{b_1 b_2}] = Z^{b_1 b_2 [a_1 \dots a_5, a_6]}, \tag{A.11}$$

The commutators with the level -1 negative root generators are given by

$$[R_{a_1 a_2 a_3}, P_b] = 0, \quad [R_{a_1 a_2 a_3}, Z^{b_1 b_2}] = 6\delta_{[a_1 a_2}^{b_1 b_2} P_{a_3]}, \quad [R_{a_1 a_2 a_3}, Z^{b_1 \dots b_5}] = \frac{5!}{2}\delta_{a_1 a_2 a_3}^{[b_1 b_2 b_3} Z^{b_4 b_5]} \tag{A.12}$$

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